Graph Colouring

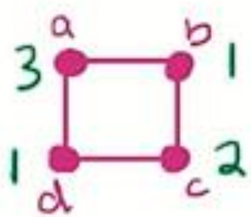
Let G be a graph with vertex set $V(G)$

A (proper) vertex colouring of G is a labelling of the vertex set

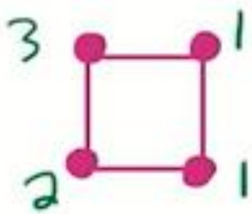
$$f: V(G) \longrightarrow \{1, 2, 3, \ldots, k\}$$

where the labels are called colours and no two adjacent vertices get the same colour.

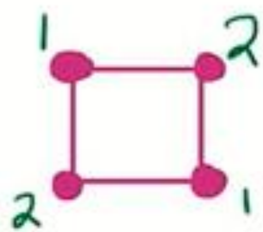
Ex:



a proper colouring with 3 colours



NOT a proper colouring



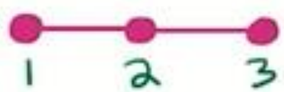
a proper colouring with 2 c

In most cases, "colouring" means "proper vertex colouring"

A k-colouring of a graph G is a colouring of G using k colours.

If G has a k-colouring then it is k-colourable

Ex:



P_3 is 3-colourable

P_3 is 2-colourable

The chromatic number of a graph G, denoted $\chi(G)$, is the smallest k such that G is k-colourable

Ex: $\chi(P_3) = 2$

If $\chi(G) = k$, we say that G is k-chromatic

Facts:

- If G has n vertices, then $\chi(G) \leq n$
- $\chi(G) = 1 \iff G$ has no edges
- $\chi(C_{2n}) = 2$ and $\chi(C_{2n+1}) = 3$
- $\chi(K_n) = n$
- If H is a subgraph of G then $\chi(G) \geq \chi(H)$

Let $G=(V,E)$

A k-colouring of G partitions the vertex set V into k sets $V_1, V_2, \ldots, V_k$ where each set V_i is an independent set

no two vertices in the set are adjacent

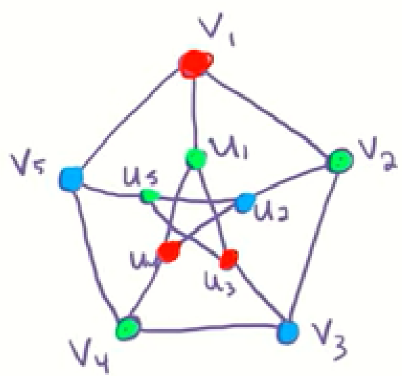
This means

$$V = V_1 \cup V_2 \cup \cdots \cup V_k$$

and $V_i \cap V_j = \emptyset$ for all $i \neq j$

The independent sets $V_1, \ldots, V_k$ are called colour classes

Ex:



colour 1
colour 2
colour 3

Colour classes:

$V_1 = \{v_1, u_3, u_4\}$

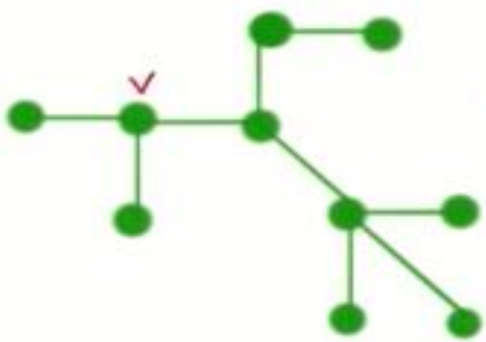
$V_2 = \{v_2, v_4, u_1, u_5\}$

$V_3 = \{v_3, v_5, u_2\}$

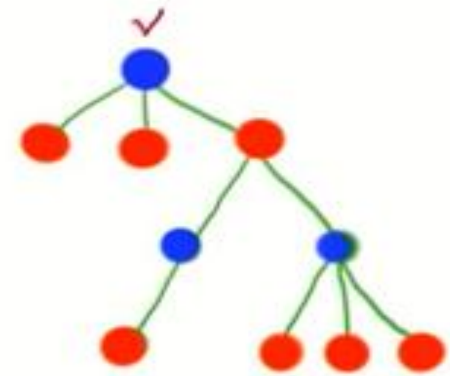
Fact: Every tree with at least 2 vertices is 2-chromatic.

Proof: Let T be a tree with $v(T) \geq 2$.

Let $v \in V(T)$ and consider T to be rooted at v.



T rooted at v:



Let v be coloured with colour 1

Let the neighbours of v be coloured with colour 2

Let the neighbours of those vertices be colour 1

$\vdots$ continue

$X = \{u \in V(T) \mid d(u,v) \text{ even}\}$

$Y = \{u \in V(T) \mid d(u,v) \text{ odd}\}$

In a tree, there is a unique path between any 2 vertices.

In a tree, there is a unique path between any 2 vertices.
Now along any path in T, the vertices alternate colours.
So no pair of adjacent vertices receive the same colour
$\therefore$ This is a 2-colouring of T and $\chi(T)=2$
since $|V(T)| \geq 2$.

We have

$$G \text{ is a tree} \implies \chi(G) = 2$$

What about the converse?

$$\chi(G) = 2 \not\Rightarrow G \text{ is a tree} \; ? \quad \text{No}$$

Ex:  $\chi(C_{2n}) = 2$

<u>Theorem</u>: $\chi(G) = 2 \iff G$ is bipartite (for non-trivial graphs)

(Recall: G is bipartite $\iff$ G has no odd cycle)

Proof:

$\Rightarrow$] Aim to show: $\chi(G) = 2 \Rightarrow G$ is bipartite

Contrapositive: G is not bipartite $\Rightarrow \chi(G) > 2$

Suppose G is not bipartite.
Then G has an odd cycle C_{2l+1} and hence $\chi(G) \geq \chi(C_{2l+1}) = 3$

✓

$\Leftarrow$] Aim to show: G is bipartite $\Rightarrow \chi(G) = 2$

Suppose G is bipartite. Then $V(G) = X \cup Y$ such that every edge of G is of the form $e = xy$ where $x \in X$ and $y \in Y$

Then colour all vertices of X colour 1 and all vertices of Y colour 2.

$\therefore \chi(G) = 2$

■

Theorem: For every graph G, $\chi(G) \le \Delta(G)+1$

Proof: By induction on n (the # of vertices)

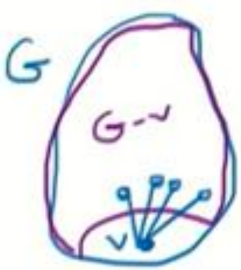
Basis: $n=1$ $\quad G = K_1$

$\chi(G) = 1$ ✓

$\Delta(G) = 0$

Ind. Hyp: Assume the result holds for every graph with $n-1$ vertices $(n \ge 2)$.

Let G be a graph on n vertices. Let $v \in V(G)$.



Now $G-v$ is a graph on $n-1$ vertices

Ind Hyp $\Rightarrow$ $\chi(G-v) \le \Delta(G-v)+1$

So $G-v$ can be coloured with at most $\Delta(G-v)+1$ colours,

Note: $\deg_G(v) \le \Delta(G)$

The neighbours of v use up $\le \Delta(G)$ colours

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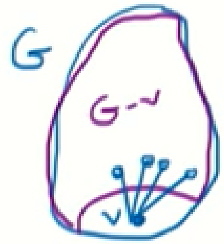

Now $G-v$ is a graph on $n-1$ vertices

Ind Hyp $\Rightarrow$ $\chi(G-v) \leq \Delta(G-v)+1$

So $G-v$ can be coloured with at most $\Delta(G-v)+1$ colours.

Note: $\deg_G(v) \leq \Delta(G)$

The neighbours of v use up $\leq \Delta(G)$ colours

Case 1: $\Delta(G) = \Delta(G-v)$

Then there is at least one colour of the $\Delta(G-v)+1 = \Delta(G)+1$ colours not used by the neighbours of v, so v can be coloured with that colour.

This gives a $(\Delta(G)+1)$-colouring of G, so $\chi(G) \leq \Delta(G)+1$ ✓

Case 2: $\Delta(G) \neq \Delta(G-v)$

Then $\Delta(G-v) < \Delta(G)$

Using a <u>new</u> colour for v we will have a $(\Delta(G-v)+2)$-colouring of G

Since $\Delta(G-v)+2 \leq \Delta(G)+1$, we have $\chi(G) \leq \Delta(G)+1$ ✓

$\chi(G) \leq \Delta(G) + 1$

Note: $\left.\begin{array}{l}\chi(K_n) = n \\ \Delta(K_n) = n-1\end{array}\right\}$ equality in the bound

$\left.\begin{array}{l}\chi(C_{2l+1}) = 3 \\ \Delta(C_{2l+1}) = 2\end{array}\right\}$ equality in the bound

Odd cycles and complete graphs are the only graphs that have equality in this bound

$\left.\begin{array}{l}G \neq K_n \\ G \neq C_{2l+1}\end{array}\right\} \Rightarrow \chi(G) \leq \Delta(G)$ Brook's Theorem