$v_1 v_3 \in E(G)$

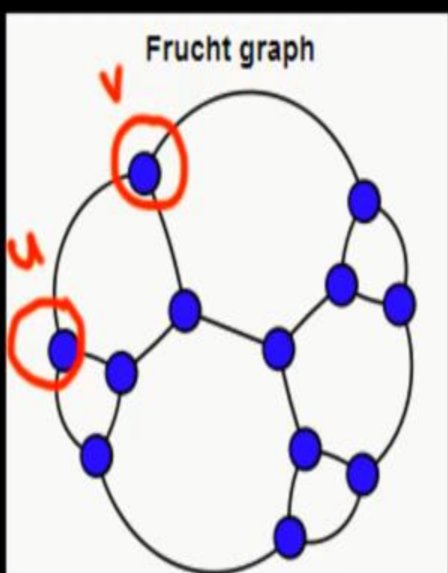
vertex-trans.

$v_3 \to v_3$

$\alpha(v_1)\alpha(v_3) = v_2 v_3 \in E(G)$

P_3

vertex-transitive graphs are <u>regular</u>



3-regular

<u>not</u> vertex-transitive

1970

Lovász Conjecture: Every finite connected vertex transitive graph has a Hamilton path.

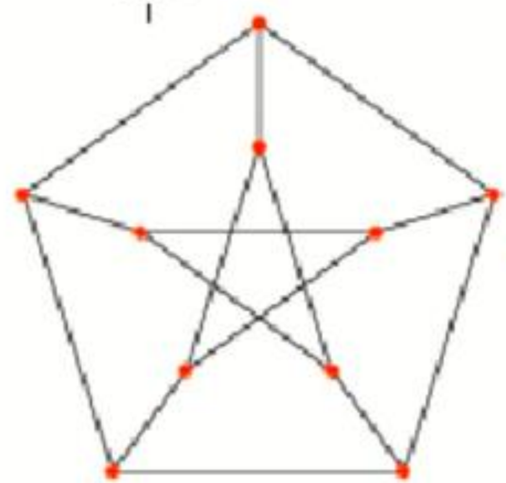
To date: no known example of a finite connected vertex-transitive graph with no Hamilton path

In fact, these almost always have a hamilton cycle.

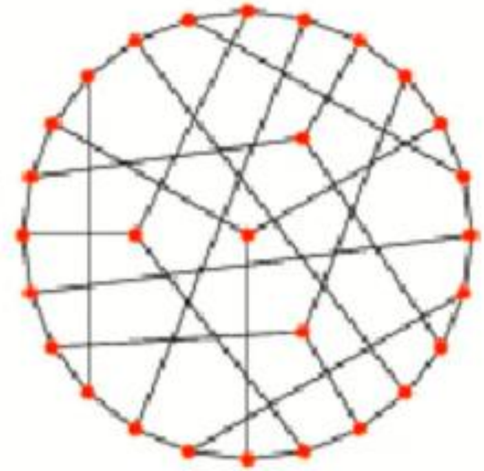
only 5 examples that don't

2-path graph

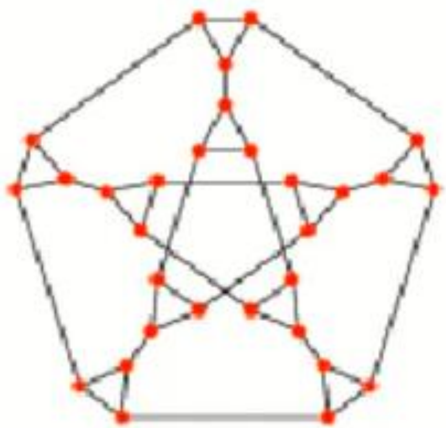
Petersen graph



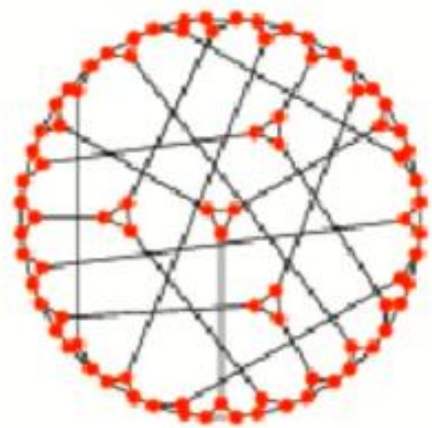
Coxeter graph



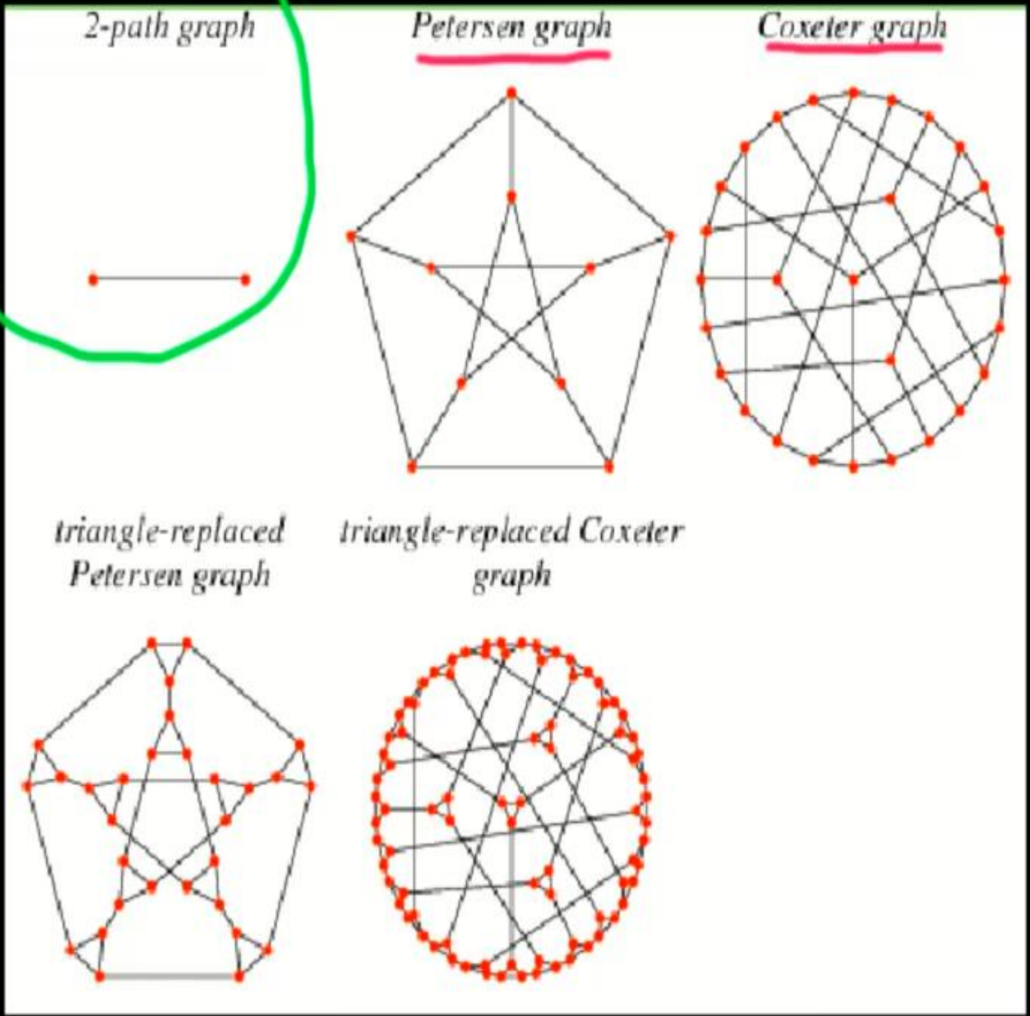
triangle-replaced Petersen graph



triangle-replaced Coxeter graph



trivial



Are there only finitely many connected vertex-transitive graph with no Hamilton cycle?

Problem

1) Find a Hamilton path in the Petersen graph

2) Prove that the Petersen graph has no Hamilton cycle.

Lemma: If G is hamiltonian, then for every nonempty subset $S \subseteq V(G)$, $\underbrace{c(G-S)}_{\text{\# of connected components of } G-S} \leq |S|$.

Proof: G is hamiltonian, let $\emptyset \neq S \subseteq V(G)$ and let $w \in S$ and $C: w \xrightarrow{*} w$ be a hamilton cycle of G.

Assume $c(G-S) = k$

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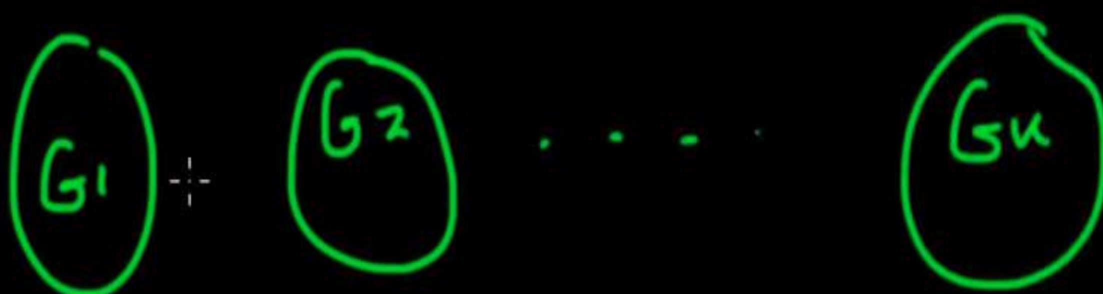
<u>Proof:</u> G is hamiltonian, let $\emptyset \neq S \subseteq V(G)$ and let $w \in S$ and $C: w \xrightarrow{*} w$ be a hamilton cycle of G.

Assume $c(G-S) = k$ and let $G_1, G_2, \ldots, G_k$ be the connected components of $G-S$.



and let $w \in S$ and $C : w \xrightarrow{*} w$ be a hamilton cycle of G.

Assume $c(G-S) = k$ and let $G_1, G_2, \ldots, G_k$ be the connected components of $G-S$.

if $k=1$ $\quad |S| \geq 1$ ✓ trivial

so $k > 1$

Give C an orientation

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Give C an orientation

Let u_i be the last vertex of C that belongs to G_i. Let v_i be the next vertex of C after u_i.

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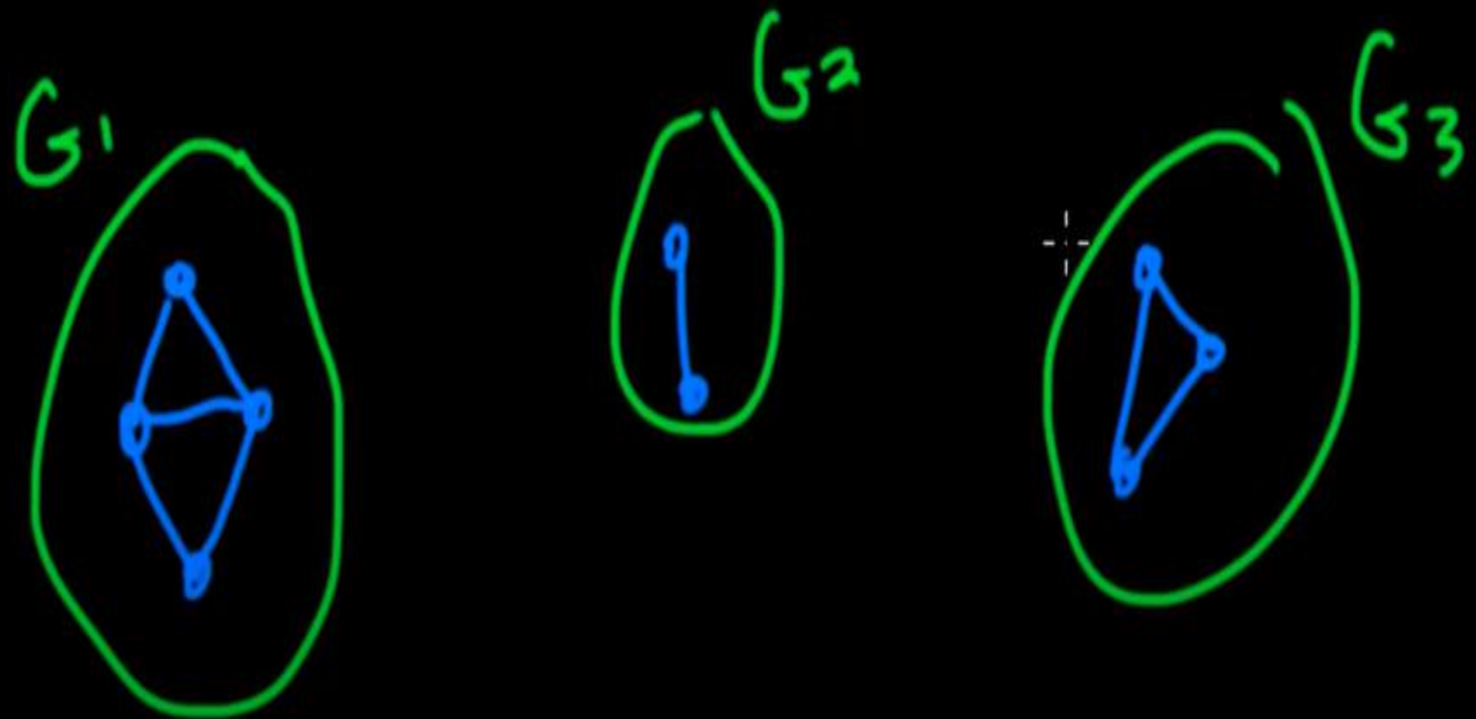
$v_i \notin V(G_i)$ (by def'n u_i)

$v_i \notin V(G_j)$ since the components are mutually disjoint

$v_i \in S$

$V_i \neq V_j$ for $i \neq j$



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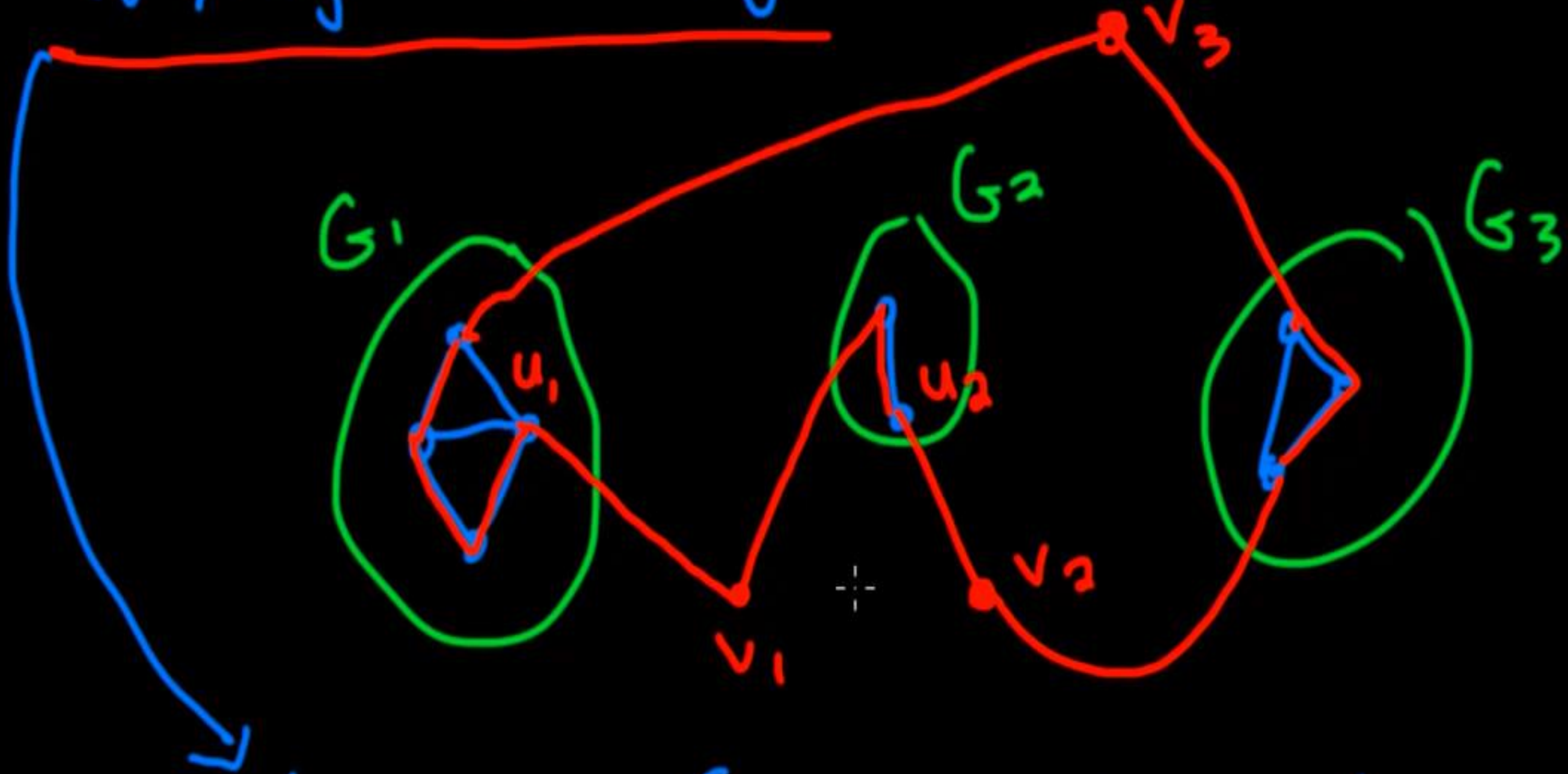
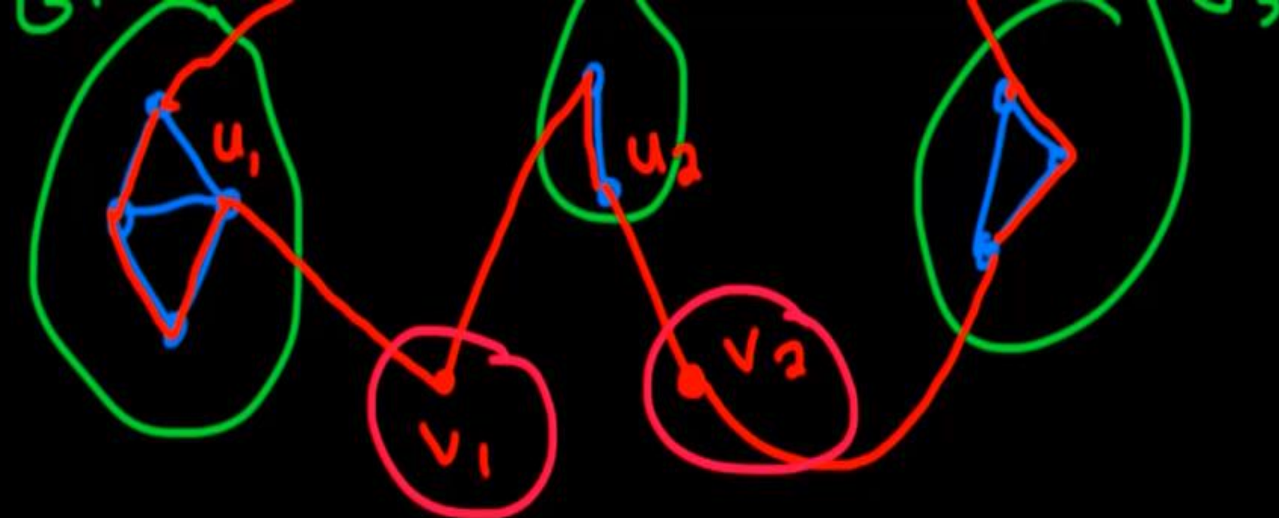


because C is a cycle and

$u_i v_i \in E(G)$ for all i



because C is a cycle and
$u_i v_i \in E(G)$ for all i

For each $i = 1, \dots, k$ there is a $v_i \in S$

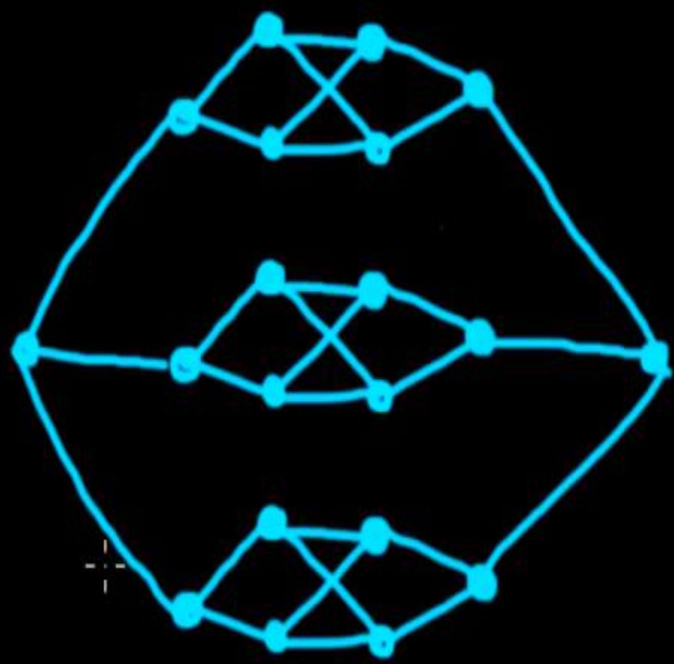
$|S| \geq k$

$$k = c(G - S) \leq |S|.$$

$\square$

Lemma: If G is hamiltonian, then for every nonempty subset $S \subseteq V(G)$, $\underbrace{c(G-S)}_{\text{\# of connected components of } G-S} \leq |S|$.

Ex:



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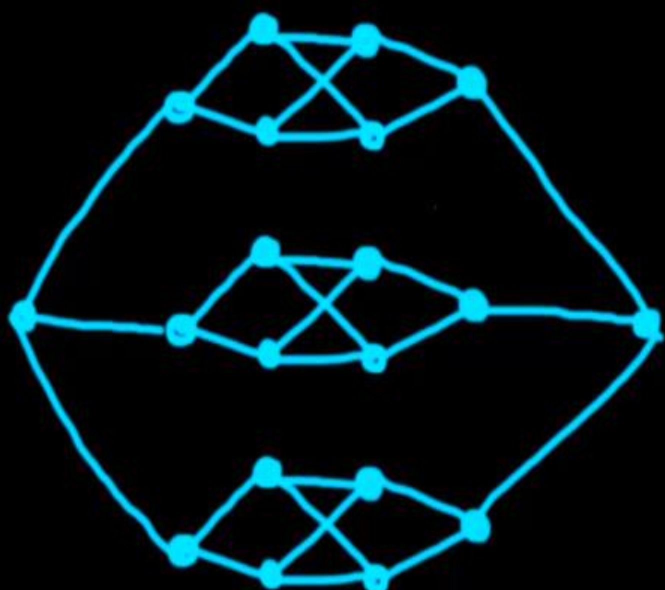

If $\exists$
$\emptyset \neq S \subseteq V(G)$
$c(G-S) > |S|$
then G is non-hamiltonian

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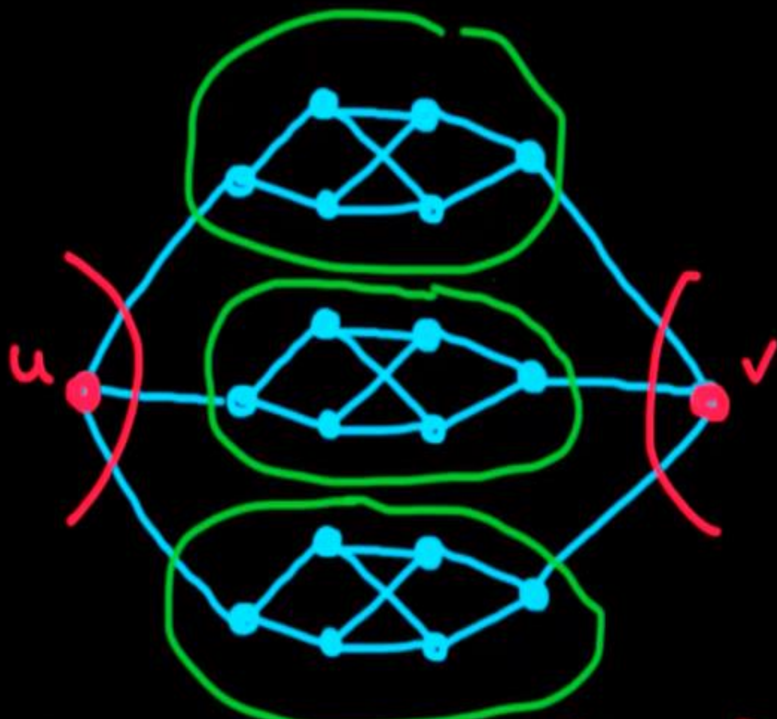

If $\exists$ $\emptyset \neq S \subseteq V(G)$ $c(G-S) > |S|$ then G is non-hamiltonian

$S = \{u, v\}$ $\quad c(G-S) = 3 \quad |S| = 2$

Ex: Petersen graph

satisfies

$\emptyset \neq S \subseteq V(G)$

$c(G-S) \leq |S|$

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However, it has no Hamilton cycle.

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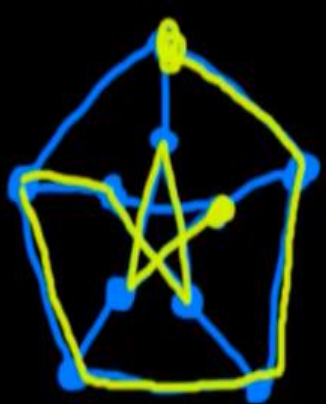
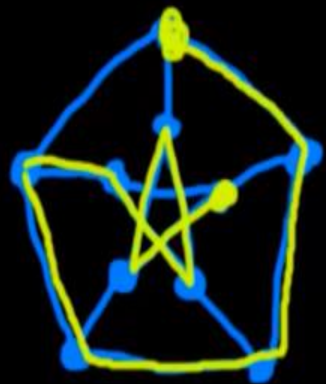


Ham. path ✓

Observe: The Petersen graph has no cycle on ≤ 4 vertices.



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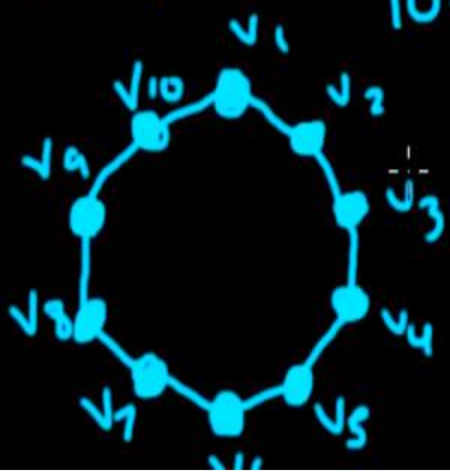
Proof (by contradiction):

Suppose C is a Hamilton cycle of the Petersen graph.

↑ 10-cycle



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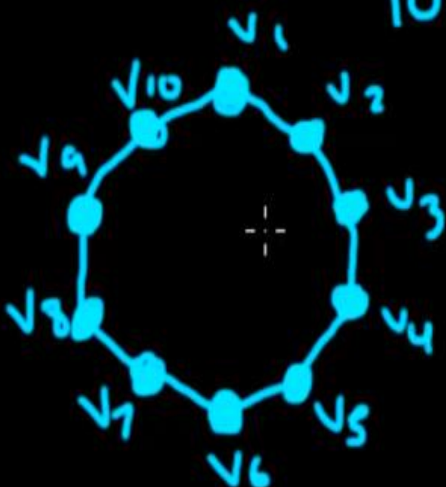
15 edges



Then the Petersen graph consists of the 10-cycle $C = (v_1, v_2, \ldots, v_{10})$ together with 5 "chord" edges that connect vertices not already adjacent in C.

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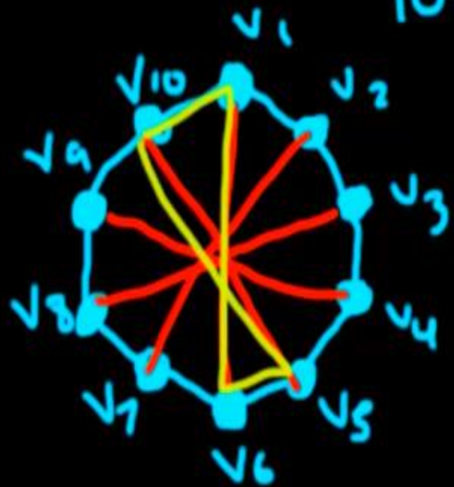

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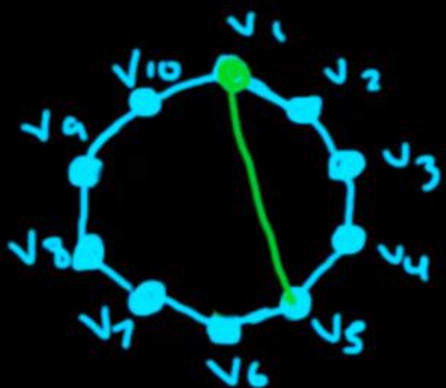

$v_1 v_5 \in E$

There is a chord that joins vertices at distance < 5 in C. These vertices must be at distance 4 in C.

If each chord
joins vertices
opposite on C
Then $\exists$ a 4-cycle. ($\Rightarrow\Leftarrow$)

edges that connect
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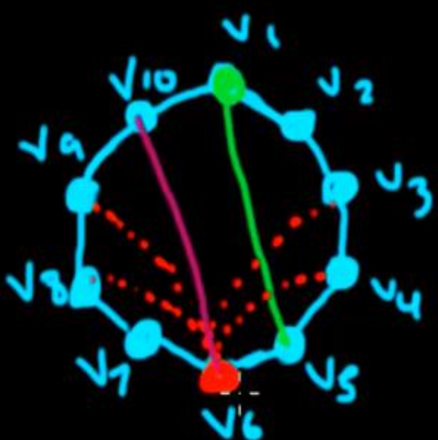

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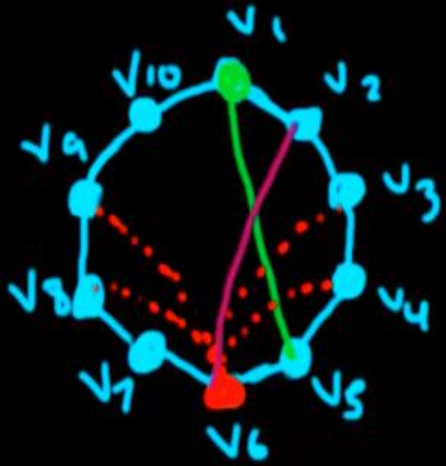

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There is a chord that joins vertices at distance < 5 in C. These vertices must be at distance 4 in C.

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Now v_6 cannot be incident with any chord edge without making a cycle of length ≤ 4. $(\Rightarrow)(\Leftarrow)$

$\square$

Thank you