Euler and Hamilton Cycles; Planar Graphs; Coloring.

Euler paths and cycles

Hamilton paths and cycles

Planar graphs

- Regions
- Euler characteristic
- Edge-Face Handshaking
- Girth

Graph Coloring

- Dual Graph
- Scheduling

Euler and Hamilton Paths -Motivation

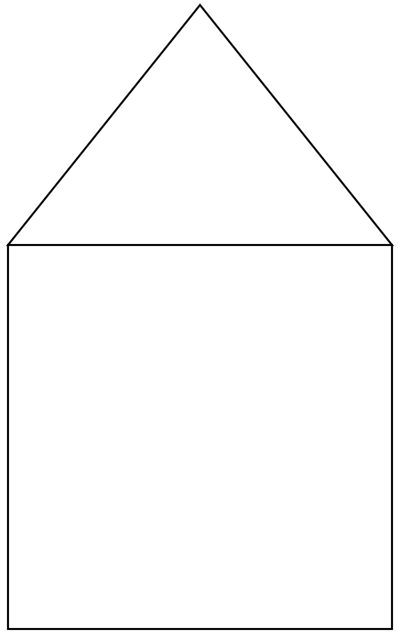
An pictorial way to motivate the graph theoretic concepts of Eulerian and Hamiltonian paths and circuits is with two puzzles:

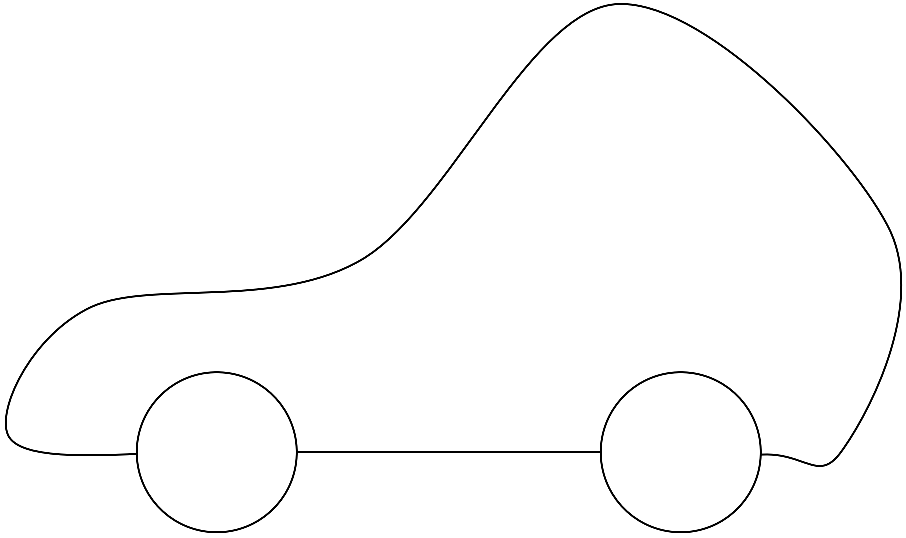
The pencil drawing problem

The taxicab problem

Pencil Drawing Problem -Euler Paths

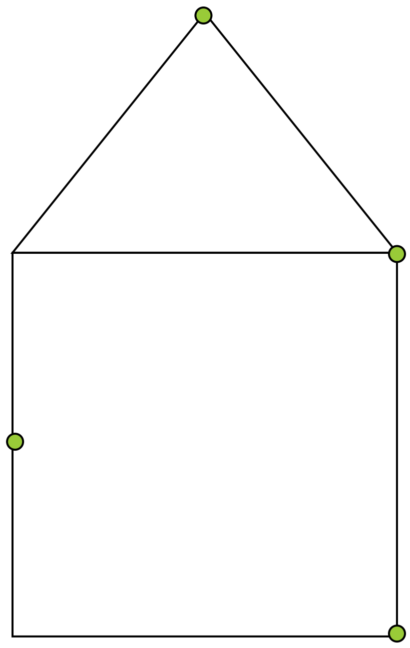
Which of the following pictures can be drawn on paper without ever lifting the pencil and without retracing over any segment?

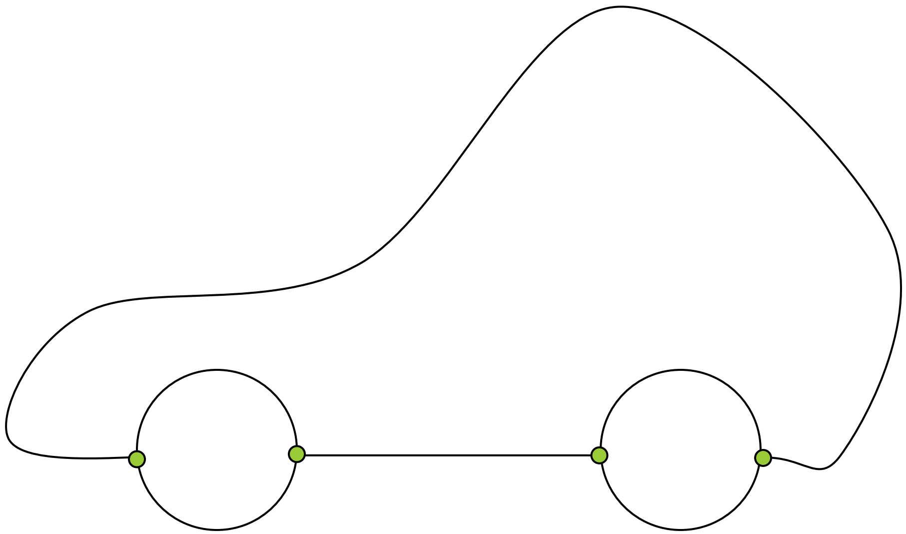




Pencil Drawing Problem -Euler Paths

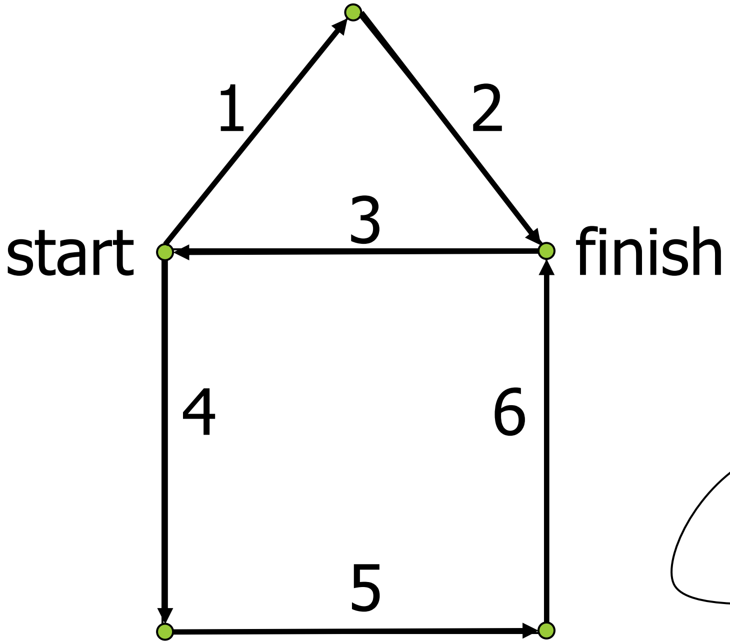
Graph Theoretically: Which of the following graphs has an Euler path?

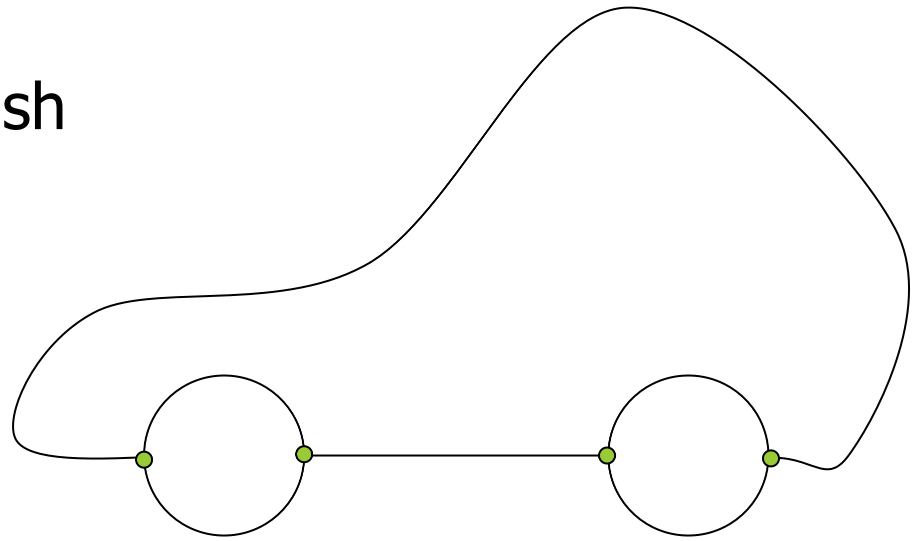




Pencil Drawing Problem
-Euler Paths

Answer: the left but not the right.





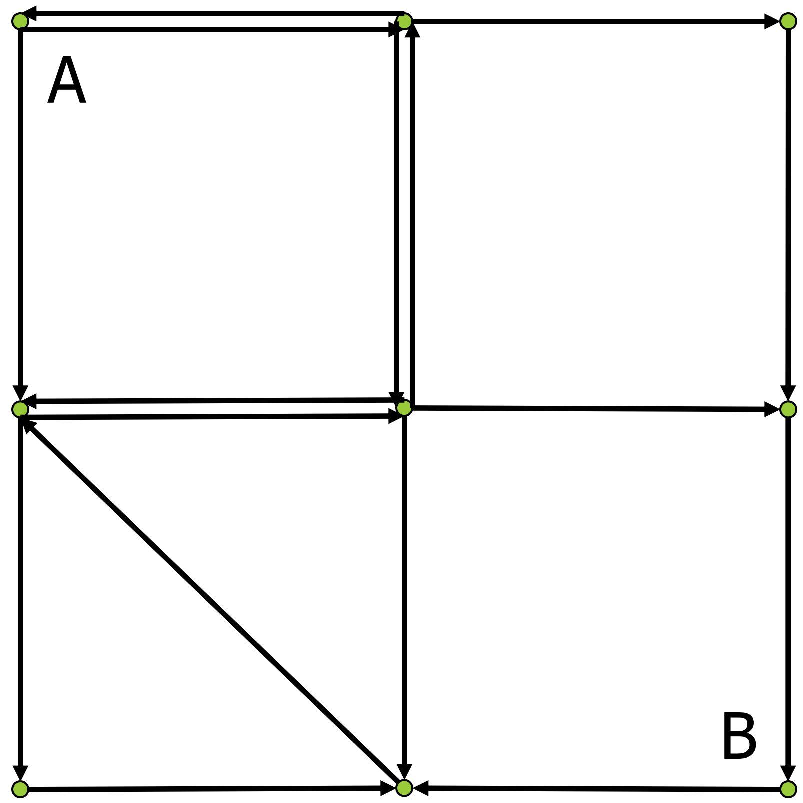
Euler Paths and Circuits Definition

DEF: An ***Euler path*** in a graph *G* is a simple path containing every edge in *G*. An ***Euler circuit*** (or ***Euler cycle***) is a cycle which is an Euler path.

NOTE: The definition applies both to undirected as well as directed graphs of all types.

Taxicab Problem -Hamilton Paths

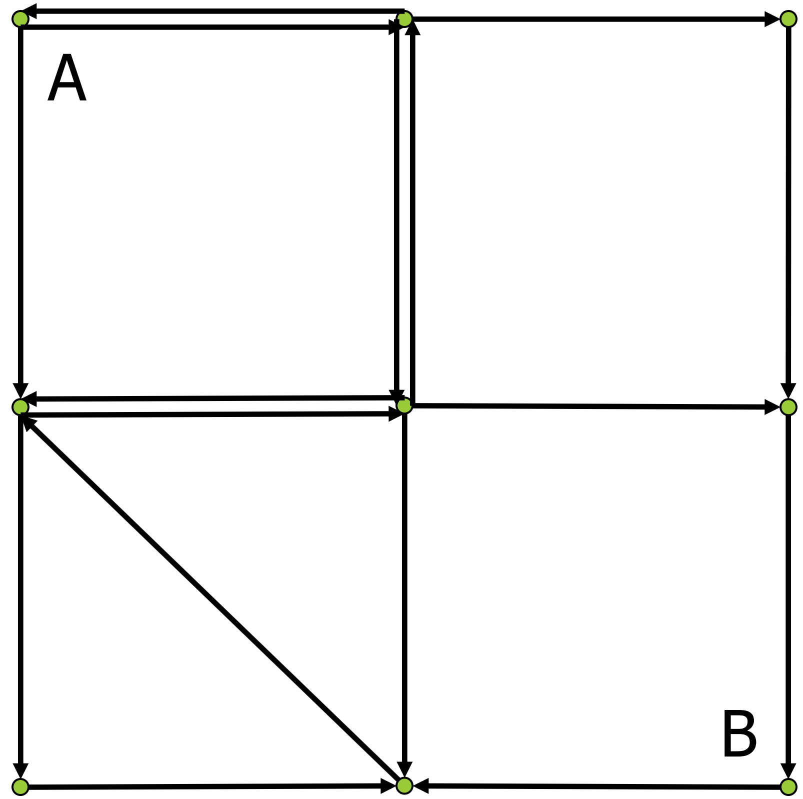
Can a taxicab driver milk his customer by visiting every intersection exactly once, when driving from point A to point B ?



Taxicab Problem -Hamilton Paths

Graph Theoretically: Is there a Hamilton path from A to B in the following graph?

(NO in this case)



Hamilton Paths and Circuits Definition

DEF: A ***Hamilton path*** in a graph *G* is a path which visits ever vertex in *G* exactly once. A ***Hamilton circuit*** (or ***Hamilton cycle***) is a cycle which visits every vertex exactly once, *except for the first vertex*, which is also visited at the end of the cycle.

NOTE: Again, the definition applies both to undirected as well as directed graphs of all types.

Implications to CS

Finding Hamilton paths is a very important problem in CS.

EG: Visit every city (vertex) in a region using the least trips (edges) as possible.

EG: Encode all bit strings of a certain length as economically as possible so that only change one bit at a time. (Gray codes).

Implications to CS

Analyzing difficulty of Euler vs. Hamilton paths is a great CS case study.

- Finding Euler paths can be done in $O(n)$ time
- Finding Hamilton paths is **NP**-complete!

Slight change in definition can result in dramatic algorithmic bifurcation!

Finding Euler Paths

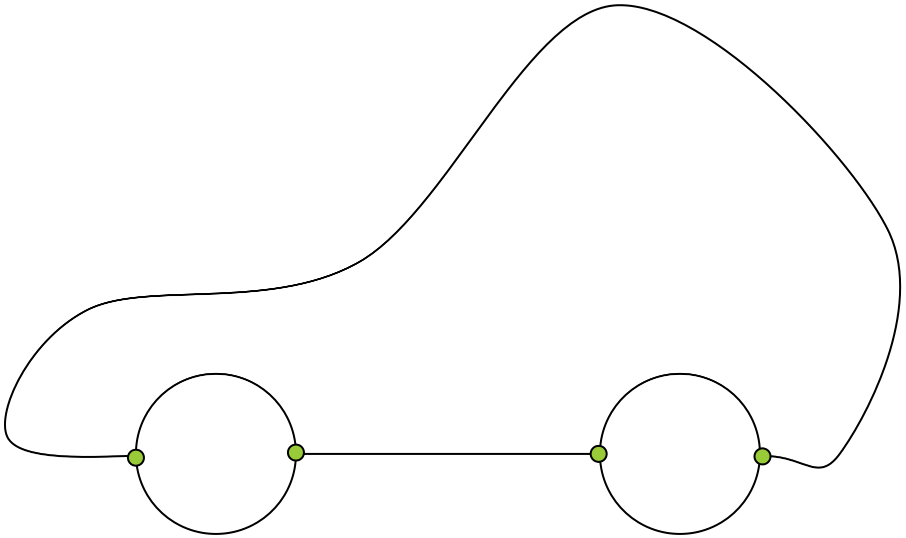
To find Euler paths, we'll first give an algorithm for finding Euler cycles and then modify it to give Euler paths.

THM: An undirected graph *G* has an Euler circuit iff it is connected and every vertex has even degree.

NOTE: for directed graphs the condition is that *G* be weakly connected and that every vertex has same in-degree as out-degree.

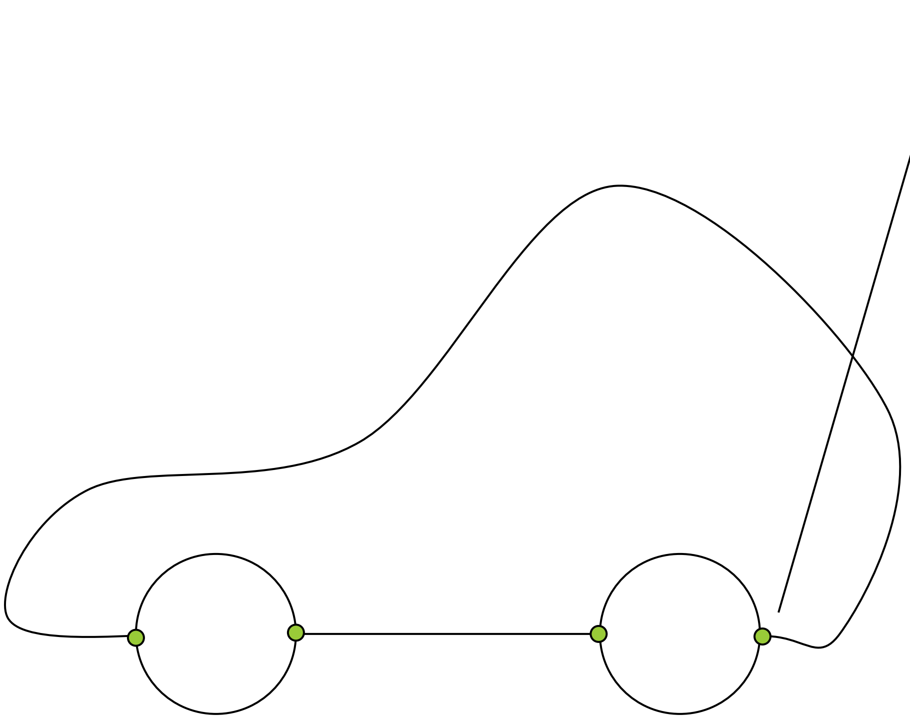
Finding Euler Circuits

Q: Why does the following graph have no Euler circuit?



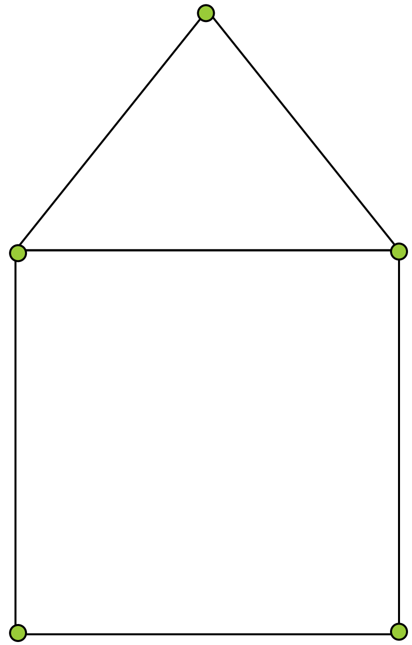
Finding Euler Circuits

A: It contains a vertex of odd degree.



Generalizing to Euler Paths

Q: Does the following have an Euler circuit?



Planar Graphs

Planar graphs are graphs that can be drawn in the plane without edges having to cross.

Understanding planar graph is important:

Any graph representation of maps/ topographical information is planar.

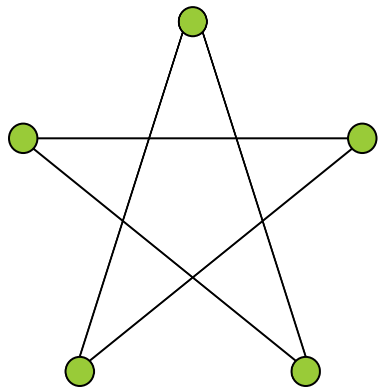
- graph algorithms often specialized to planar graphs (e.g. traveling salesperson)

Circuits usually represented by planar graphs

Planar Graphs -Common Misunderstanding

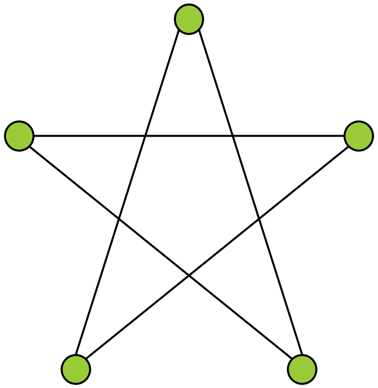
Just because a graph is drawn with edges crossing doesn't mean its not planar.

Q: Why can't we conclude that the following is non-planar?



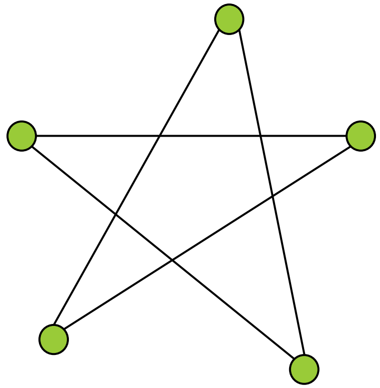
Planar Graphs -Common Misunderstanding

A: Because it is isomorphic to a graph which *is* planar:



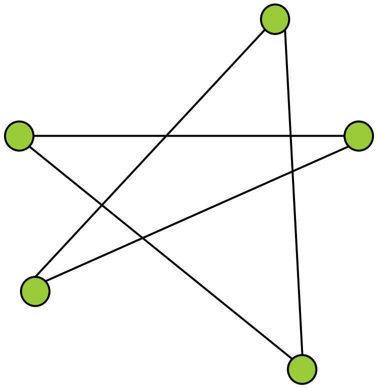
Planar Graphs
-Common Misunderstanding

A: Because it is isomorphic to a graph which *is* planar:



Planar Graphs
-Common Misunderstanding

A: Because it is isomorphic to a graph which *is* planar:



Planar Graphs -Common Misunderstanding

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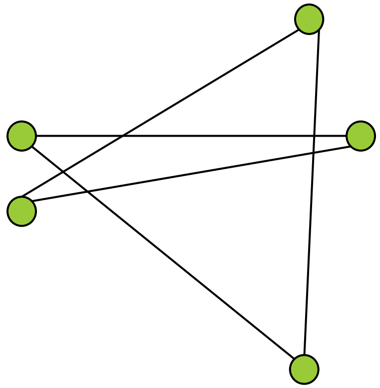

Planar Graphs
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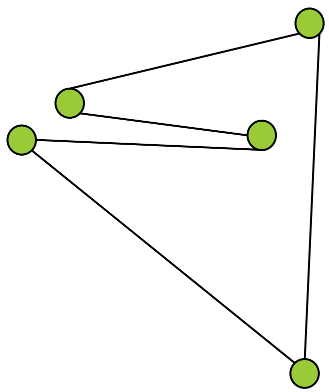

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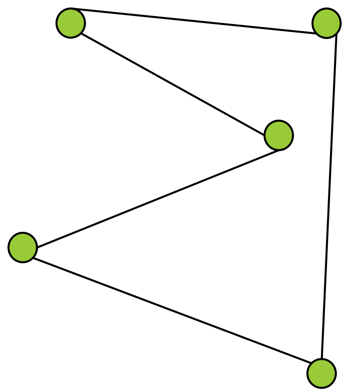

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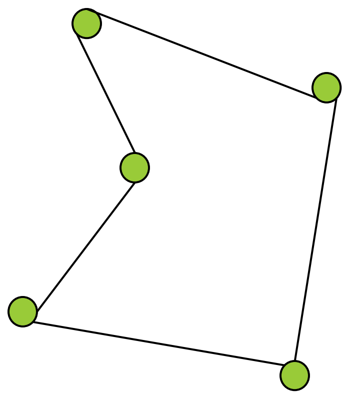
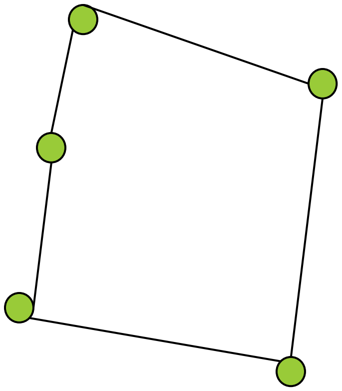


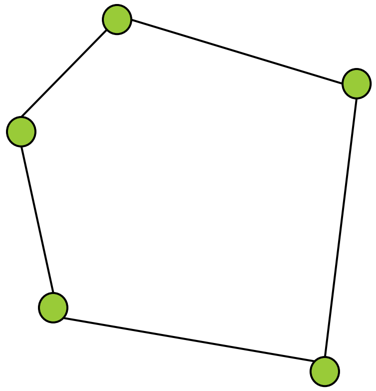
Planar Graphs
-Common Misunderstanding

A: Because it is isomorphic to a graph which *is* planar:



Planar Graphs -Common Misunderstanding

A: Because it is isomorphic to a graph which *is* planar:

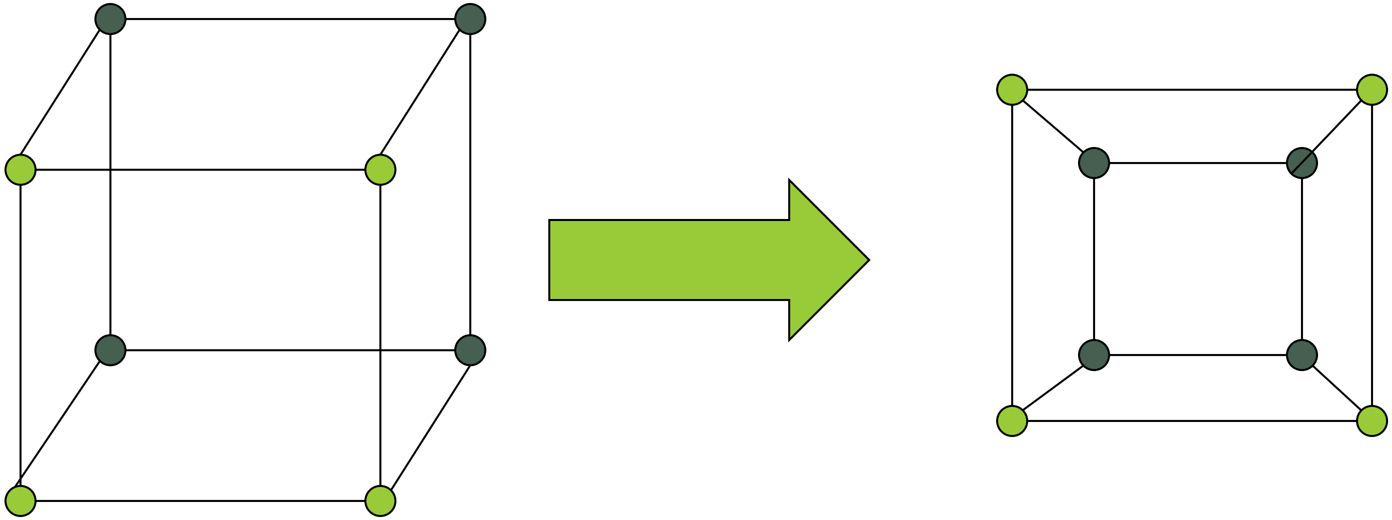


Proving Planarity

To prove that a graph is planar amounts to redrawing the edges in a way that no edges will cross. May need to move vertices around and the edges may have to be drawn in a very indirect fashion.

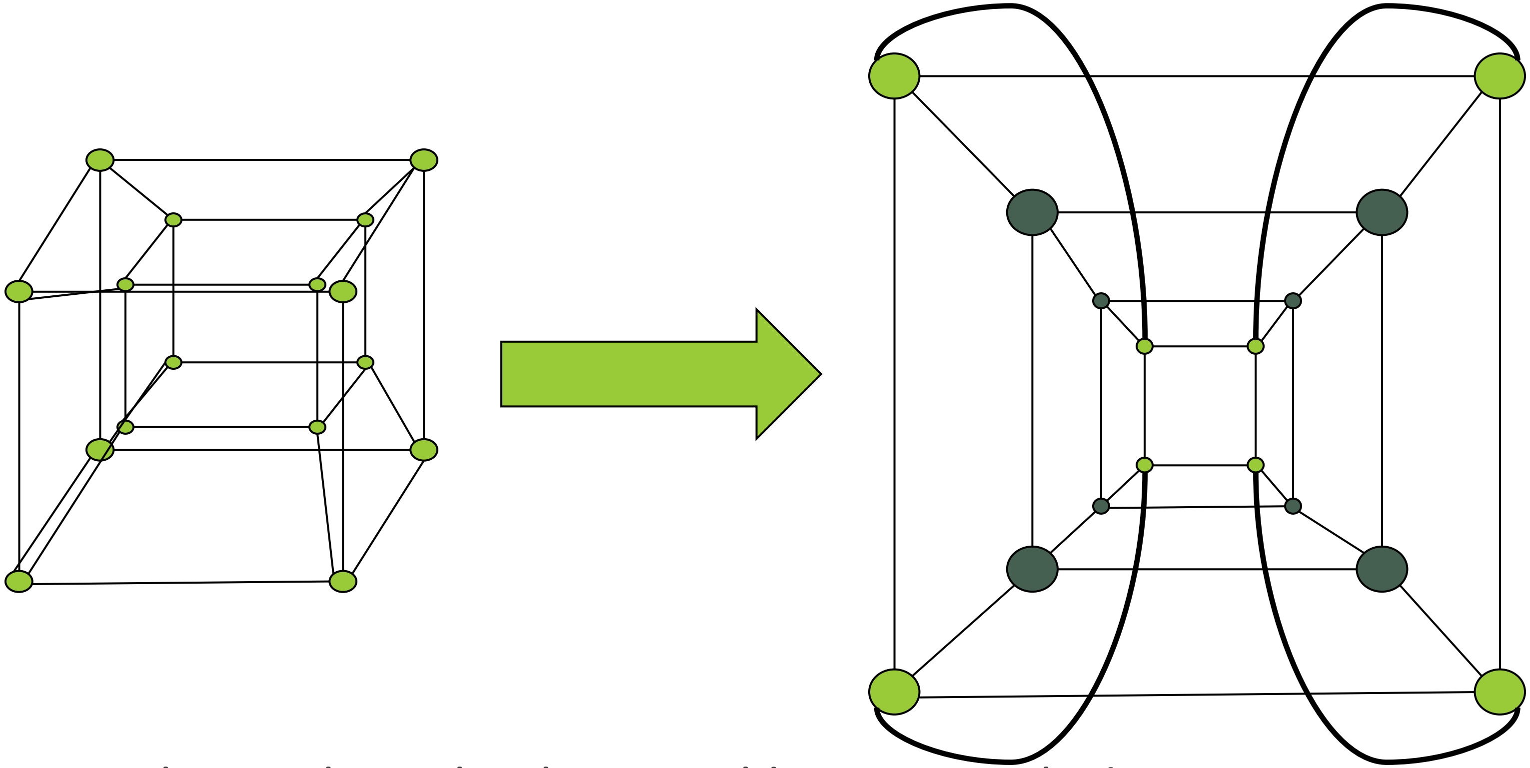
E.G. show that the 3-cube is planar:

Proving Planarity
3-Cube



Proving Planarity?

4-Cube



Seemingly not planar, but how would one prove this!

Disproving Planarity

The book gives several methods. I'll describe one method that will always work in examples that you'll get on the final. You may also use any of the methods that the book mentions.

(One method –Kuratowski's theorem– in principle always works, though in practice can be quite unwieldy.)

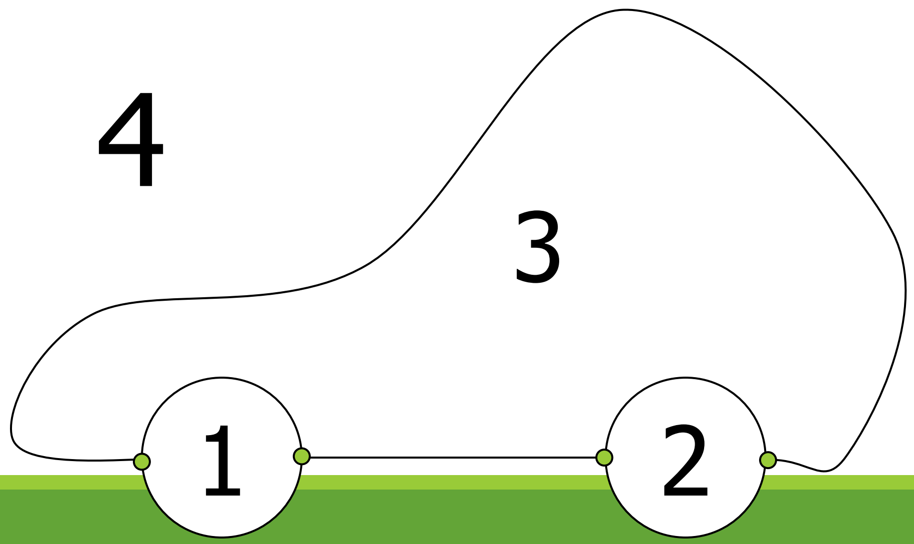
Disproving Planarity

The idea is to try to find some *invariant* quantities possessed by graphs which are constrained to certain values, for planar graphs. Then to show that a graph is non-planar, compute the quantities and show that they do not satisfy the constraints on planar graphs.

Regions

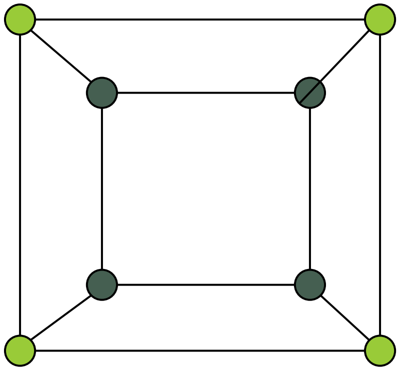
The first invariant of a planar graph will be the number of ***regions*** that the graph defines in the plane. A region is a part of the plane completely disconnected off from other parts of the plane by the edges of the graph.

EG: the car graph has 4 regions:



Regions

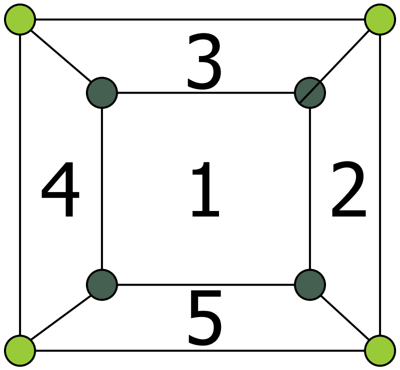
Q: How many regions does the 3-cube have?



Regions

A: 6 regions

6



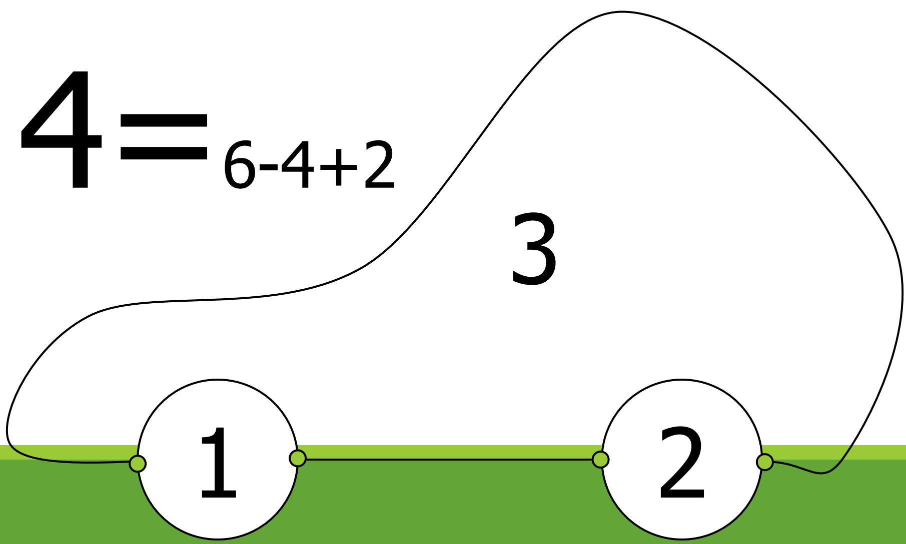
Regions

THM: The number of regions defined by a connected planar graph is invariant of how it is drawn in the plane and satisfies the formula involving edges and vertices:

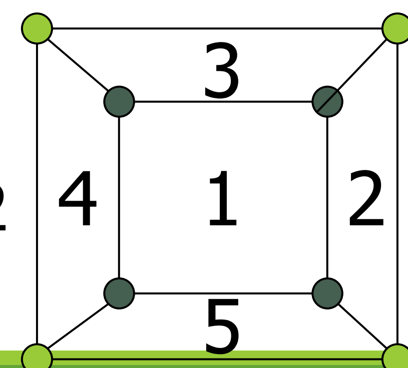
$$r = |E| - |V| + 2$$

EG: Verify formula for car and 3-cube:

$4 =_{6-4+2}$



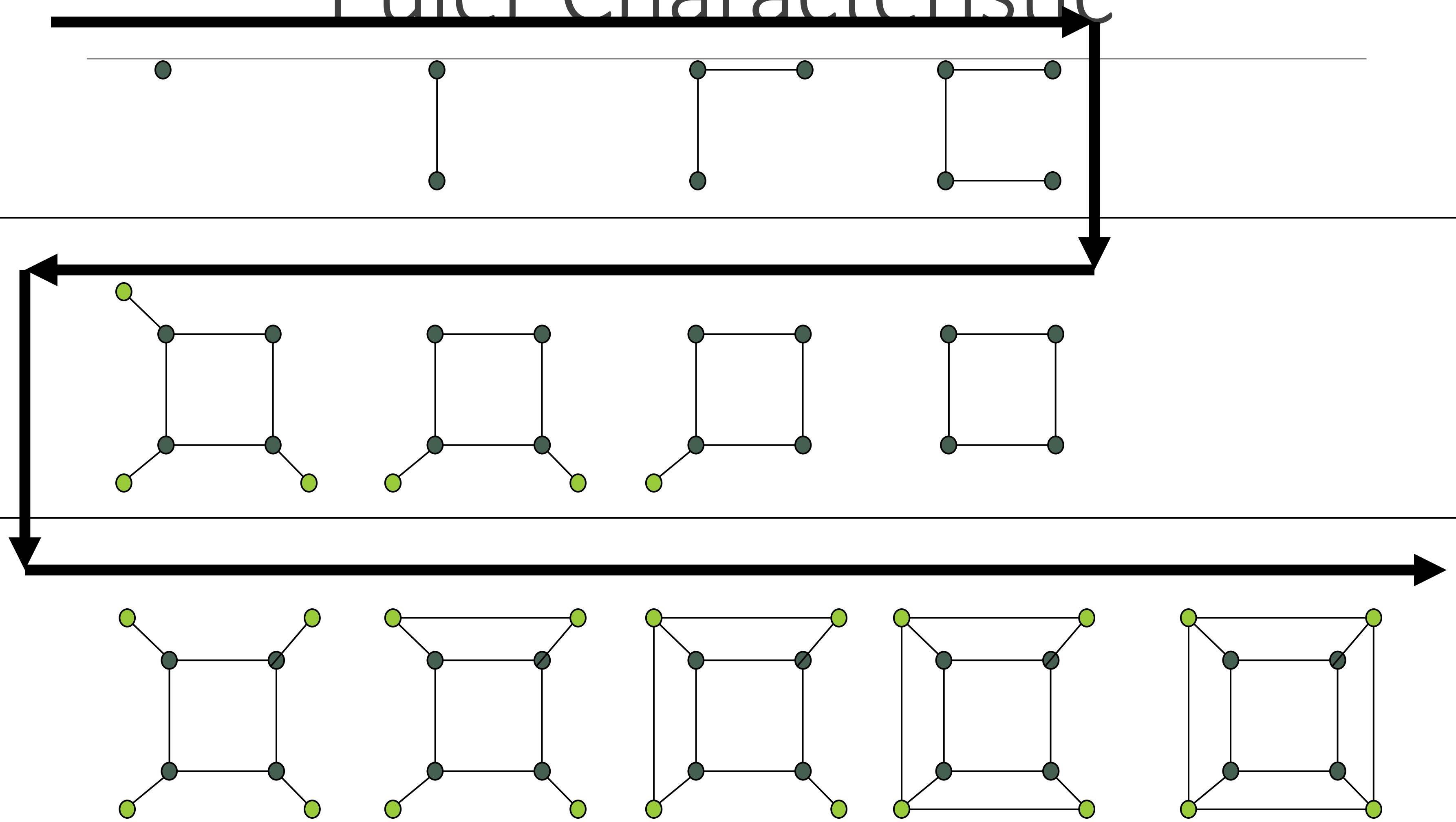
$6 =_{12-8+2}$



Euler Characteristic

The formula is proved by showing that the quantity (chi) $\chi = r - |E| + |V|$ must equal 2 for planar graphs. χ is called the ***Euler characteristic***. The idea is that any connected planar graph can be built up from a vertex through a sequence of vertex and edge additions. For example, build 3-cube as follows:

Euler Characteristic



Euler Characteristic

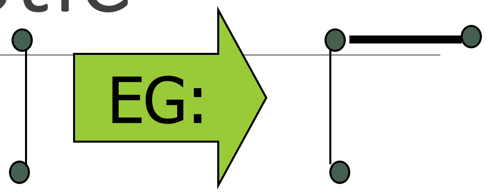
Thus to prove that χ is always 2 for planar graphs, one calculate χ for the trivial vertex graph:

$$\chi = 1-0+1 = 2$$

and then checks that each possible move does not change χ .

Euler Characteristic

Check that moves don't change χ :

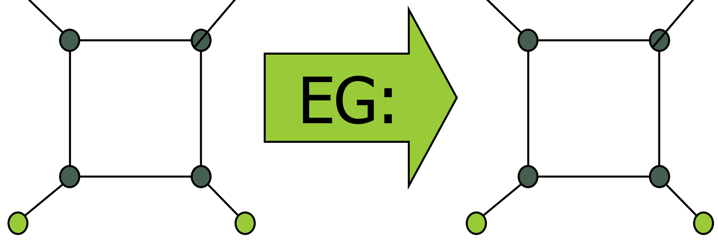


1) Adding a degree 1 vertex:

r is unchanged. |*E* | increases by 1. |*V* | increases by 1. χ += (0-1+1)

2) Adding an edge between pre-existing vertices:

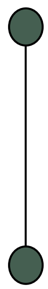
r increases by 1. |*E* | increases by 1. |*V* | unchanged. χ += (1-1+0)



Animated Invariance of Euler Characteristic

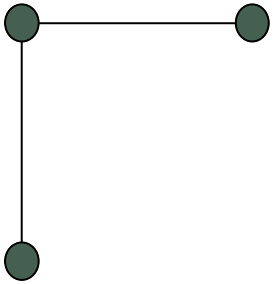
$\lvert V \rvert$	$\lvert E \rvert$	r	$\chi = r - \lvert E \rvert + \lvert V \rvert$
1	0	1	2

Animated Invariance of Euler Characteristic



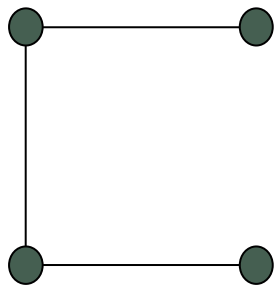
$\|V\|$	$\|E\|$	r	$\chi = r - \|E\| + \|V\|$
2	1	1	2

Animated Invariance of Euler Characteristic



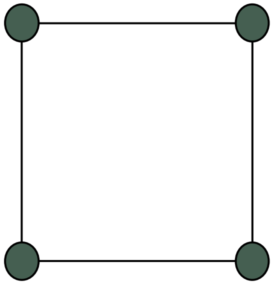
$\lvert V \rvert$	$\lvert E \rvert$	r	$\chi = r - \lvert E \rvert + \lvert V \rvert$
3	2	1	2

Animated Invariance of Euler Characteristic



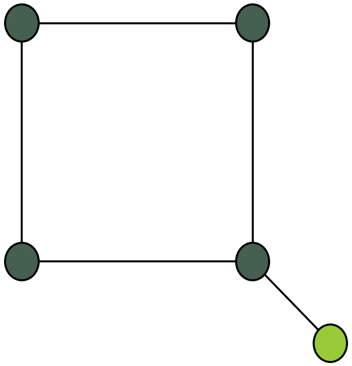
$\vert V \vert$	$\vert E \vert$	r	$\chi = r - \vert E \vert + \vert V \vert$
4	3	1	2

Animated Invariance of Euler Characteristic



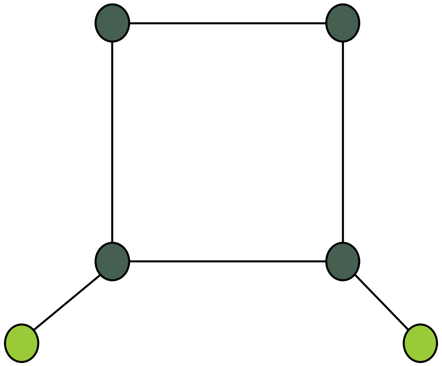
$\|V\|$	$\|E\|$	r	$\chi = r - \|E\| + \|V\|$
4	4	2	2

Animated Invariance of Euler Characteristic



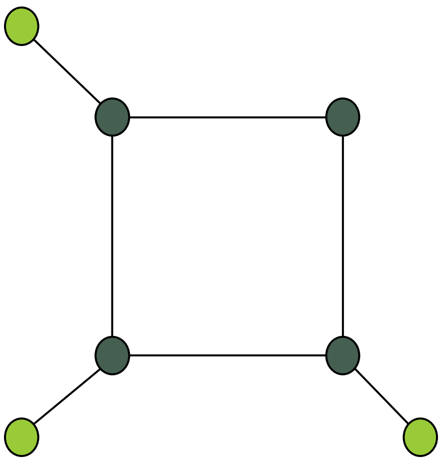
$\|V\|$	$\|E\|$	r	$\chi = r - \|E\| + \|V\|$
5	5	2	2

Animated Invariance of Euler Characteristic



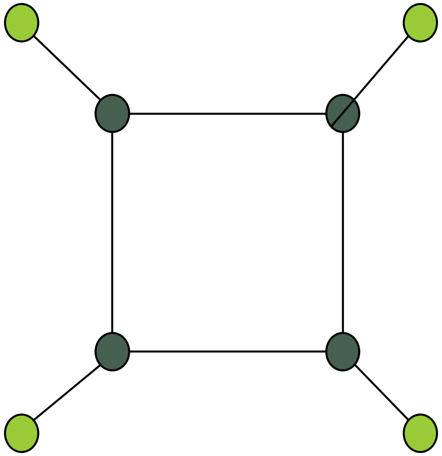
$\lvert V \rvert$	$\lvert E \rvert$	r	$\chi = r - \lvert E \rvert + \lvert V \rvert$
6	6	2	2

Animated Invariance of Euler Characteristic



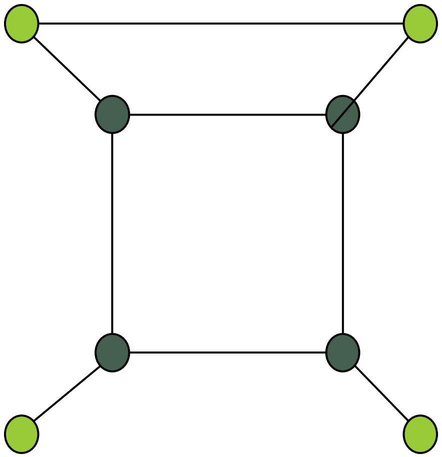
$\vert V \vert$	$\vert E \vert$	r	$\chi = r - \vert E \vert + \vert V \vert$
7	7	2	2

Animated Invariance of Euler Characteristic



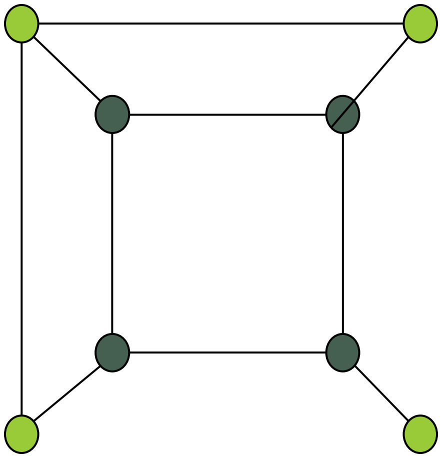
$\|V\|$	$\|E\|$	r	$\chi =$ $r - \|E\| + \|V\|$
8	8	2	2

Animated Invariance of Euler Characteristic



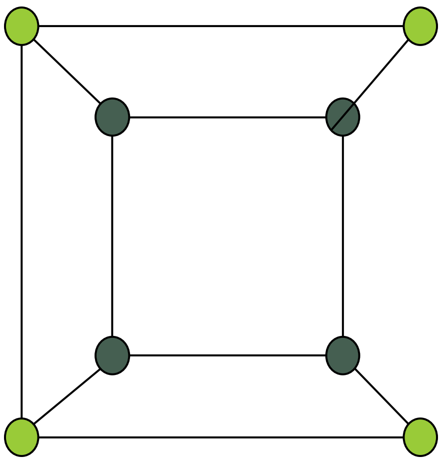
$\|V\|$	$\|E\|$	r	$\chi = r - \|E\| + \|V\|$
8	9	3	2

Animated Invariance of Euler Characteristic



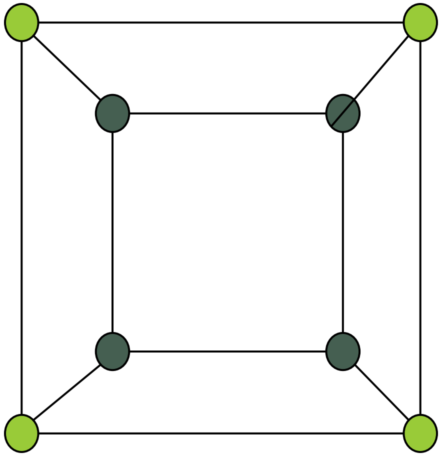
$\|V\|$	$\|E\|$	r	$\chi = r - \|E\| + \|V\|$
8	10	4	2

Animated Invariance of Euler Characteristic



$\|V\|$	$\|E\|$	r	$\chi =$ $r - \|E\| + \|V\|$
8	11	5	2

Animated Invariance of Euler Characteristic



$\|V\|$	$\|E\|$	r	$\chi = r - \|E\| + \|V\|$
8	12	6	2

Face-Edge Handshaking

For **all** graphs handshaking theorem relates degrees of *vertices* to number of *edges*.

For **planar** graphs, can relate *regions* to *edges* in similar fashion:

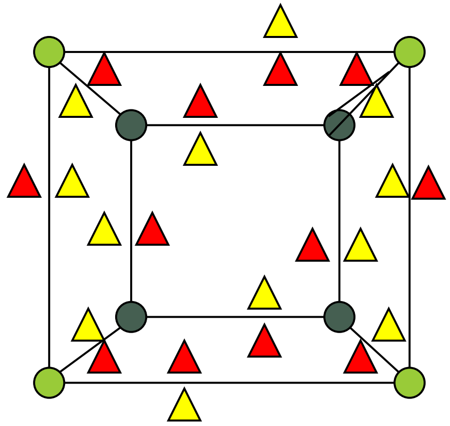
EG: There are two ways to count the number of edges in 3-cube:

1) Count directly: 12

2) Count no. of edges around

each region; divide by 2:

(4+4+4+4+4+4)/2 = 12 (2 *triangles per edge*)



Face-Edge Handshaking

DEF: The ***degree** of a **region*** *F* is the number of edges at its boundary, and is denoted by deg(*F*).

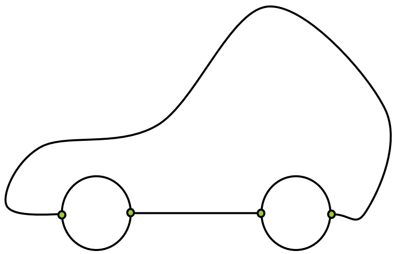
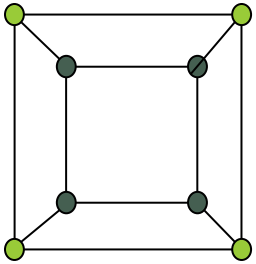
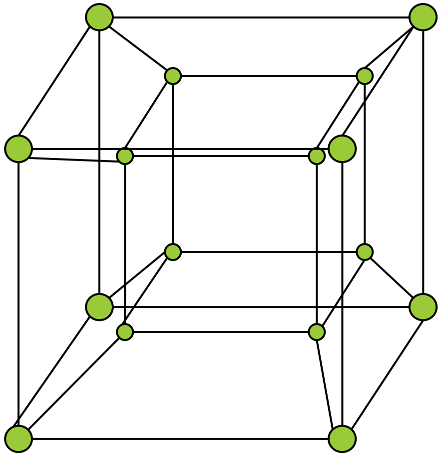
THM: Let *G* be a planar graph with region set *R*. Then:

$$|E| = \frac{1}{2}\sum_{F \in R} \deg(F)$$

Girth

The ***girth*** of a graph is the length of the smallest simple cycle in the graph.

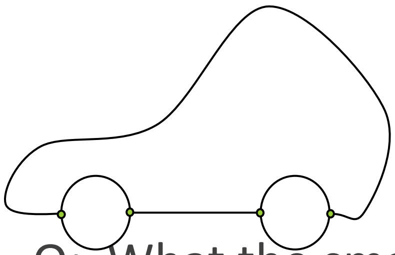
Q: What the girth of each of the following?

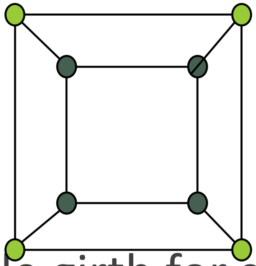




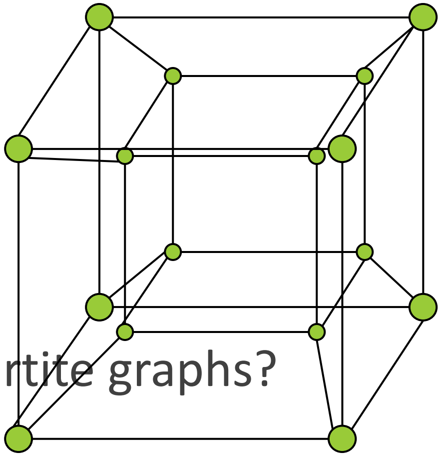
Girth

A:

$g = 2$ $g = 4$ $g = 4$







Q: What the smallest possible girth for simple bipartite graphs?

Girth

A: $g = 4$ is the smallest possible girth: Any cycle must start and end in the same color, so must have even length. Since simple, cannot have a 2-cycle, so 4-cycle is shortest possible.

Proving that Q_4 is Non-Planar

Now we have enough invariants to prove that the 4-cube is non-planar.

1) Count the number of vertices and edges:

$|V| = 16$ (twice the number for 3-cube)

$|E| = 32$ (twice the number for 3-cube plus number of vertices in 3-cube)

2) Suppose 4-cube were planar so by Euler's formula number of regions would be:

$$r = 32\text{-}16+2=18$$

Proving that Q_4 is Non-Planar

3) Calculate the girth: $g = 4$

4) Apply handshaking theorem to get a lower bound on the number of edges, since the degree of each face must be at least as large as the girth:

$$|E| = \frac{1}{2}\sum_{F \in R} \deg(F) \geq \frac{1}{2}\sum_{F \in R} g = \frac{1}{2}rg$$

In our case, this give $|E| \geq \frac{1}{2} \cdot 18 \cdot 4 = 36$

contradicting $|E| = 32$!

Thus 4-cube cannot be planar. ?

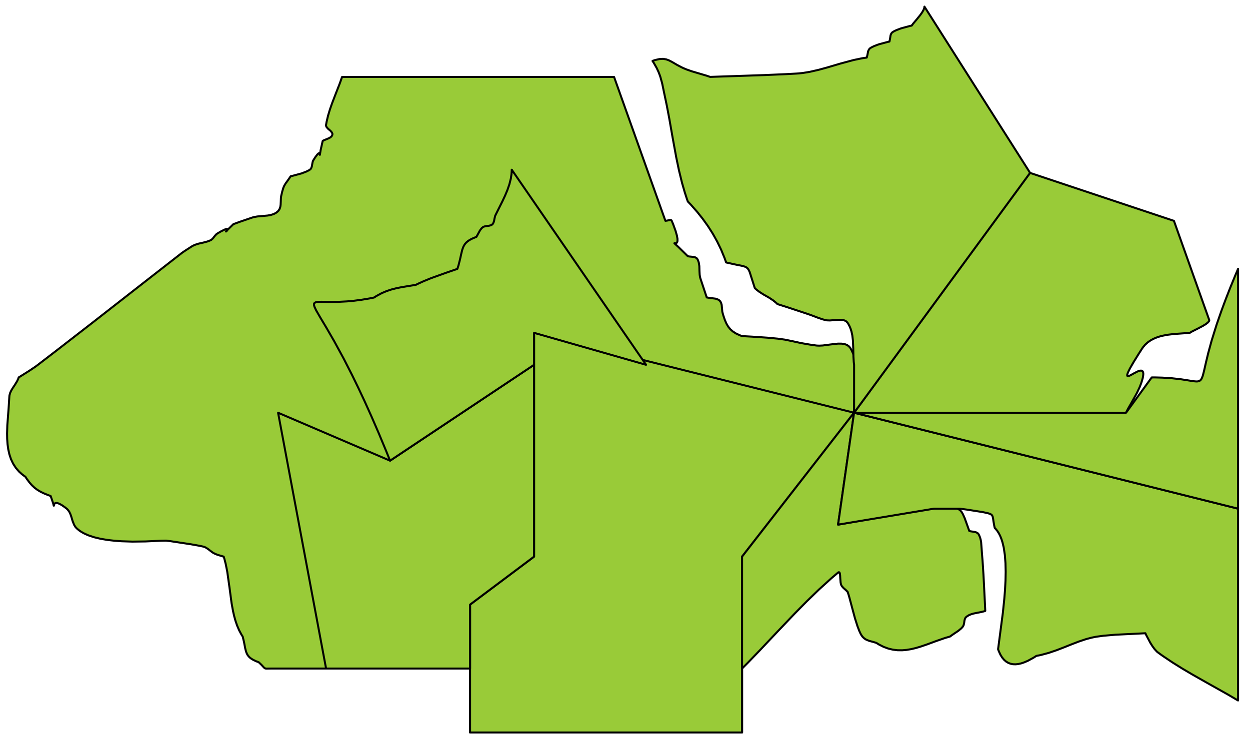
Blackboard exercises for 7.7

Show that the following graphs are non-planar:

1) K_5
2) $K_{3,3}$
3) Q_n for $n \geq 4$

Graph Coloring

Consider a fictional continent.



Map Coloring

Suppose removed all borders but still wanted to see all the countries. 1 color insufficient.

Map Coloring

So ~~add another color. Try to fill in every country with one of the two colors.~~

Map Coloring

~~So add another color. Try to fill in every country with one of the two colors.~~

Map Coloring

~~So add another color. Try to fill in every country with one of the two colors.~~

Map Coloring

So ~~add another color. Try to fill in every country with one of the two colors.~~

Map Coloring

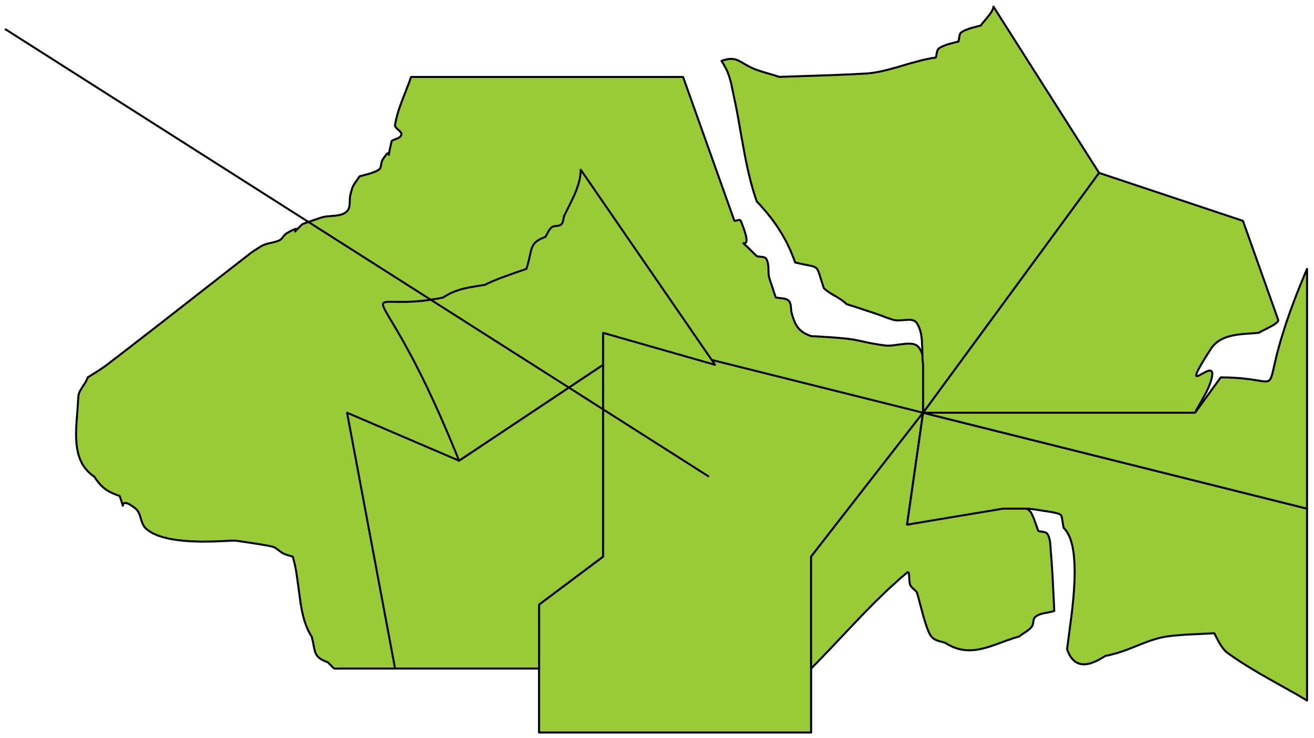
PROBLEM: Two adjacent countries forced to have same color. Border unseen.

Map Coloring

So add another color:

Map Coloring

Insufficient. Need 4 colors because of <u>this</u> country.



Map Coloring

With 4 colors, could do it.

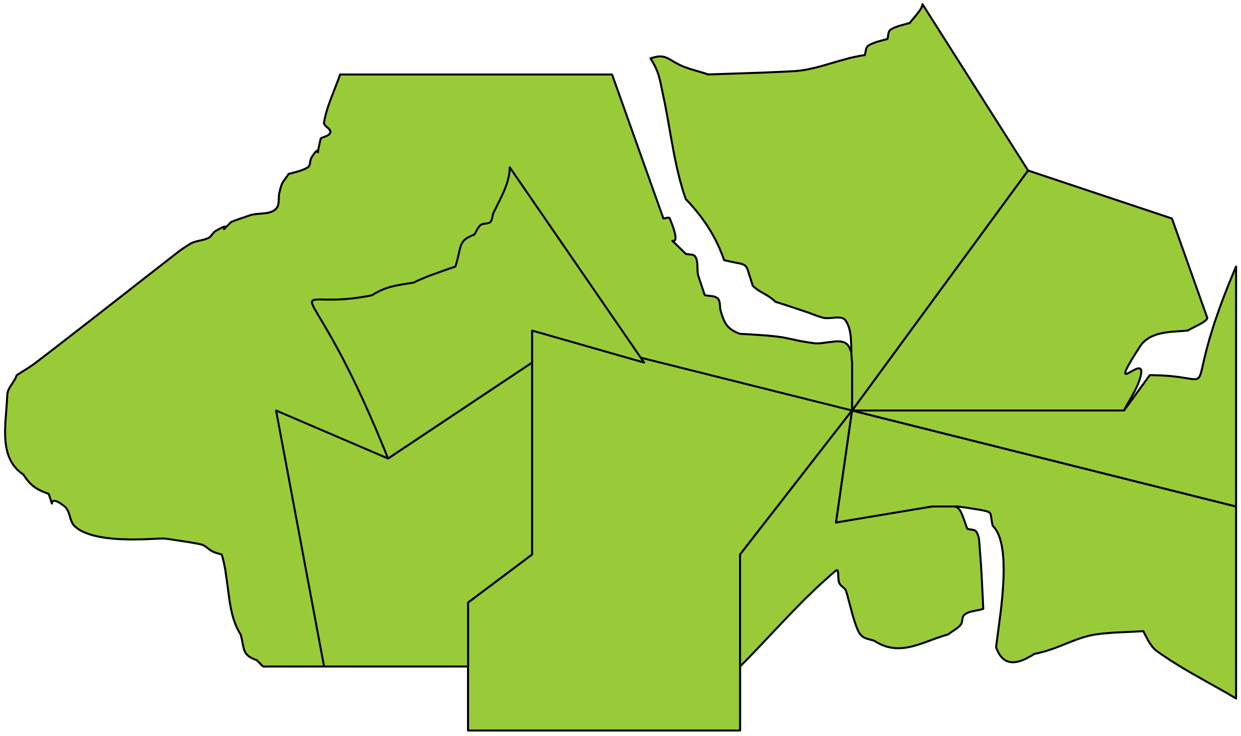
4-Color Theorem

THM: Any planar map of regions can be depicted using 4 colors so that no two regions that share a positive-length border have the same color.

Proof by Haaken and Appel used exhaustive computer search.

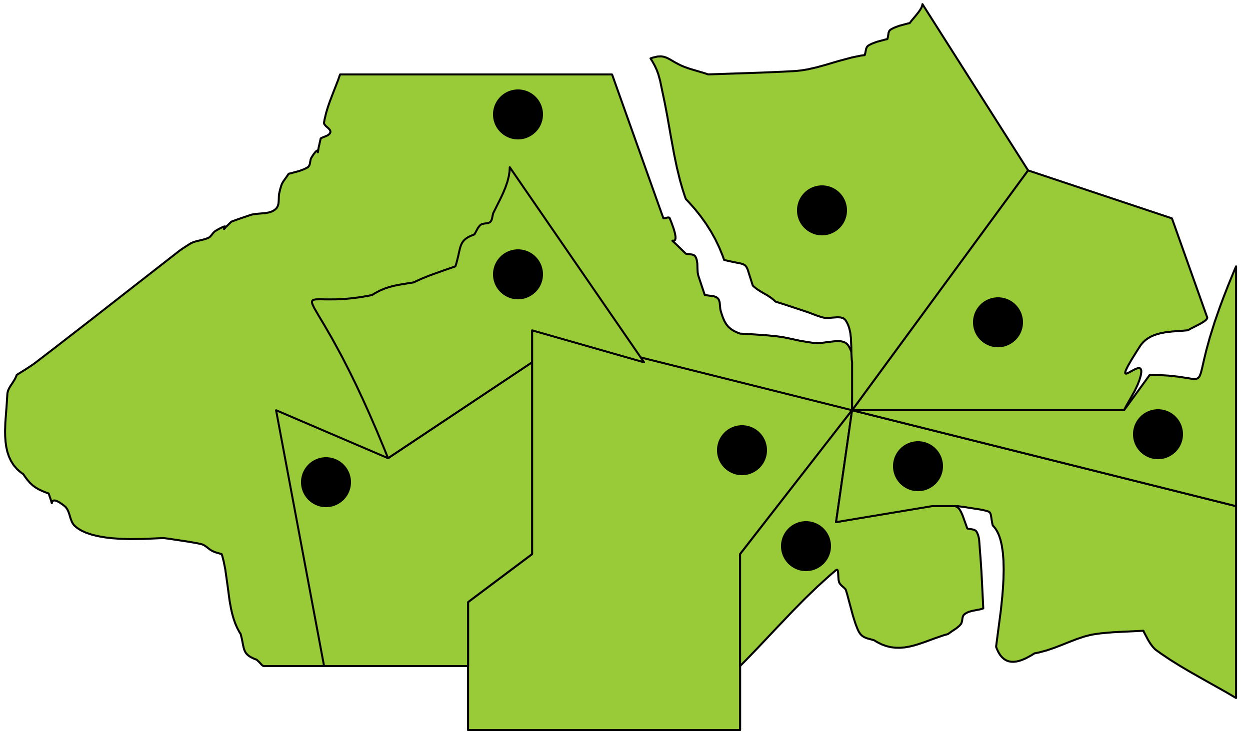
From Map Coloring to Graph Coloring

The problem of coloring a map, can be reduced to a graph-theoretic problem:



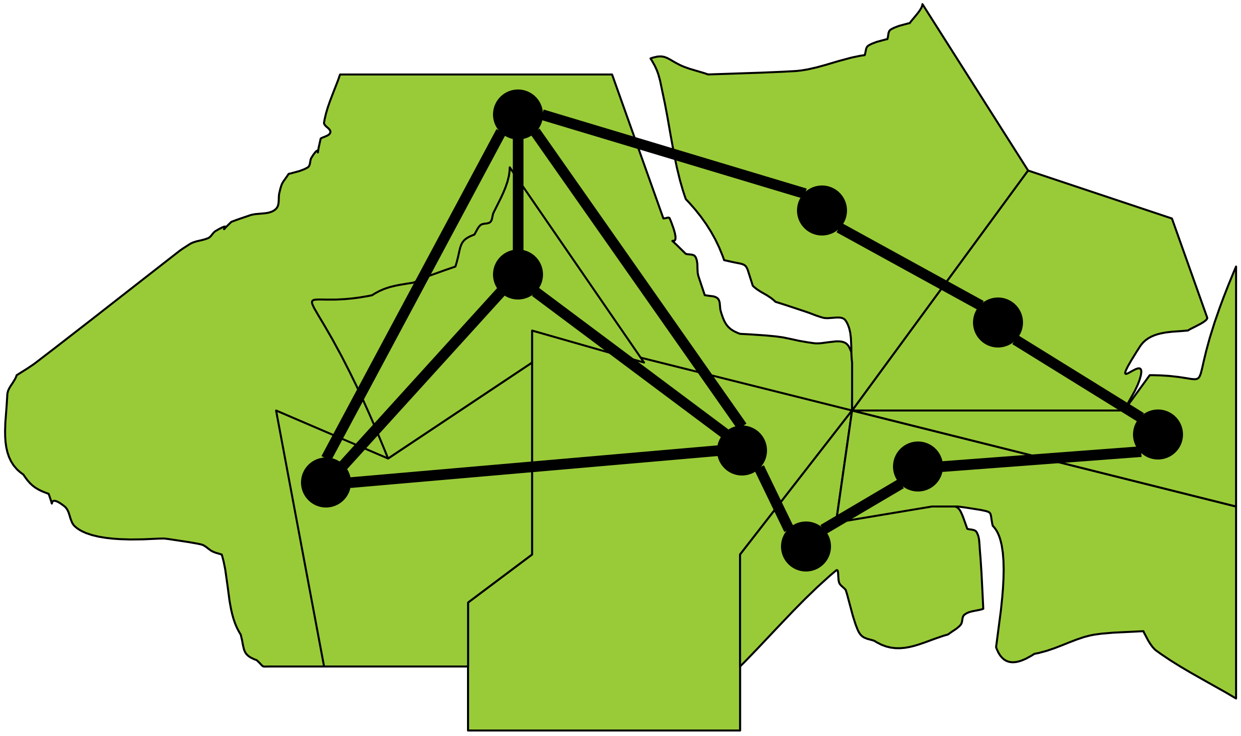
From Map Coloring to Graph Coloring

For each region introduce a vertex:



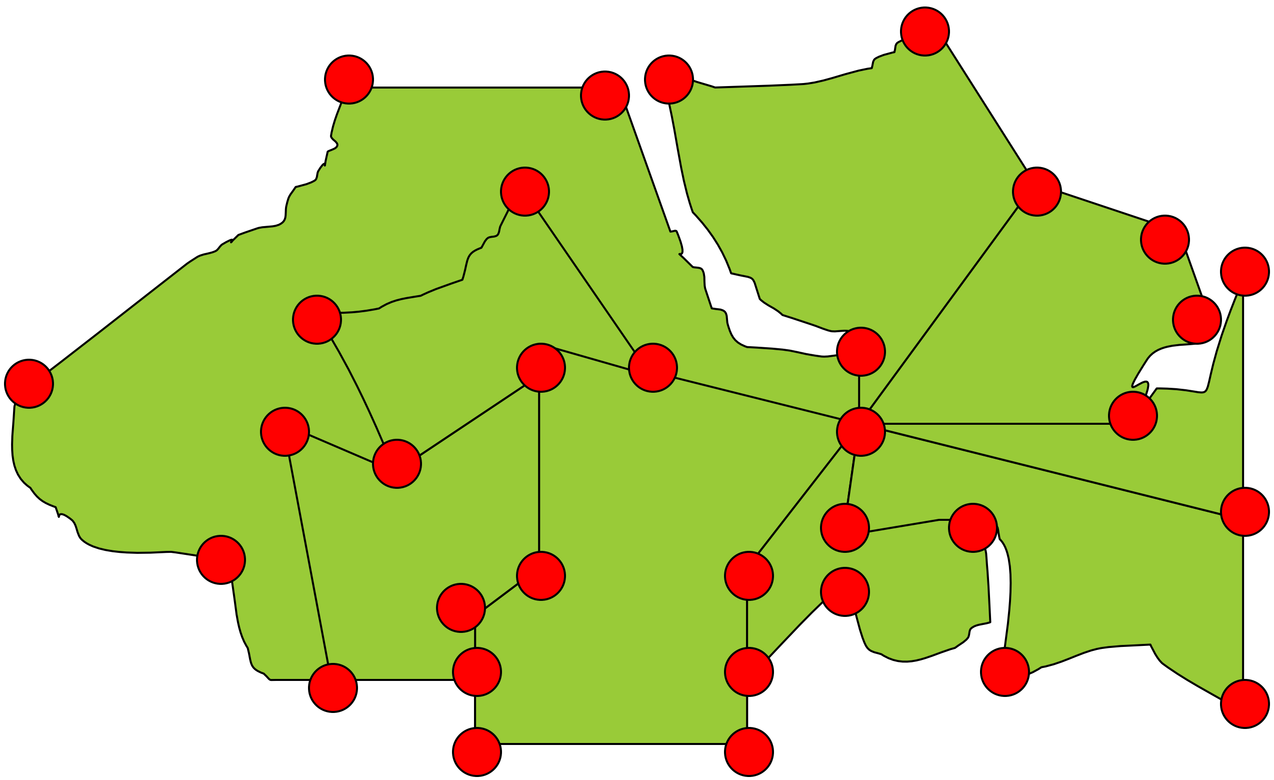
From Map Coloring to Graph Coloring

For each pair of regions with a positive-length common border introduce an edge:



From Maps to Graphs to Dual Graphs

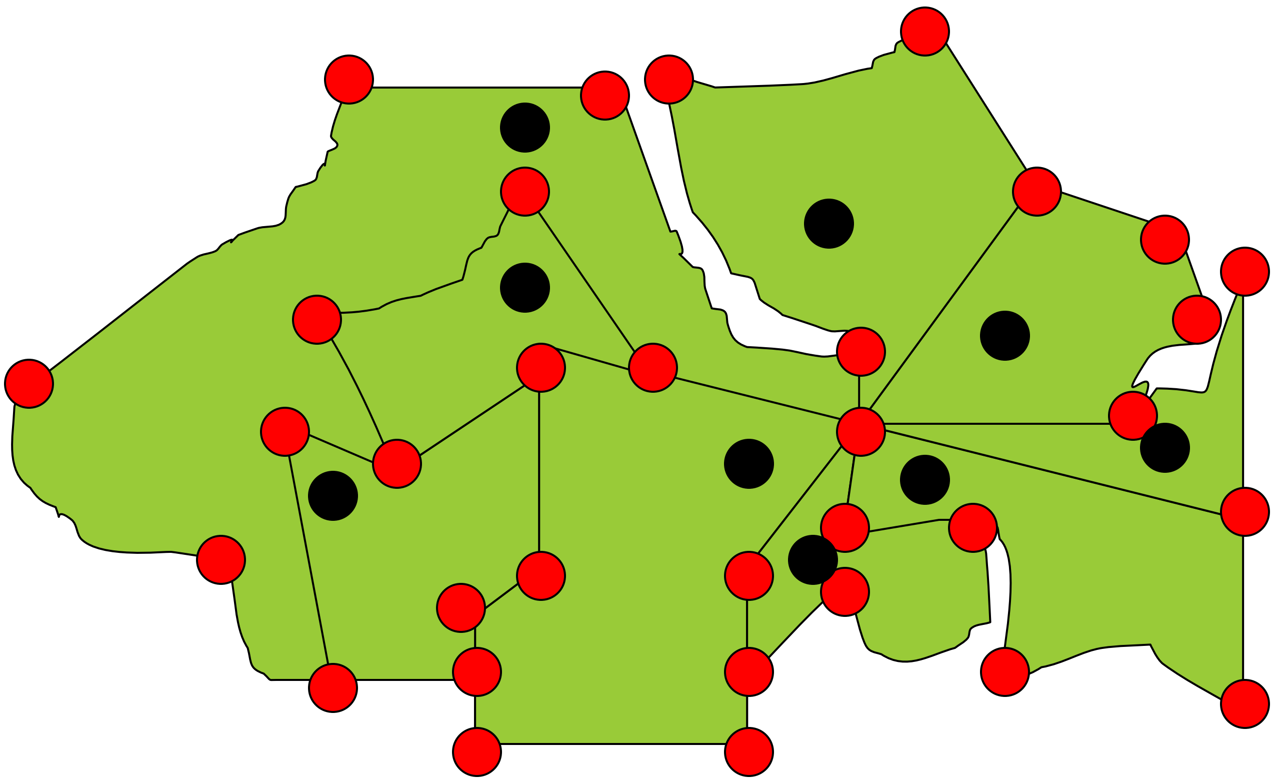
Really, could think of original map as a graph, and we are looking at *dual graph*:



From Maps to Graphs to Dual Graphs

Dual Graphs :

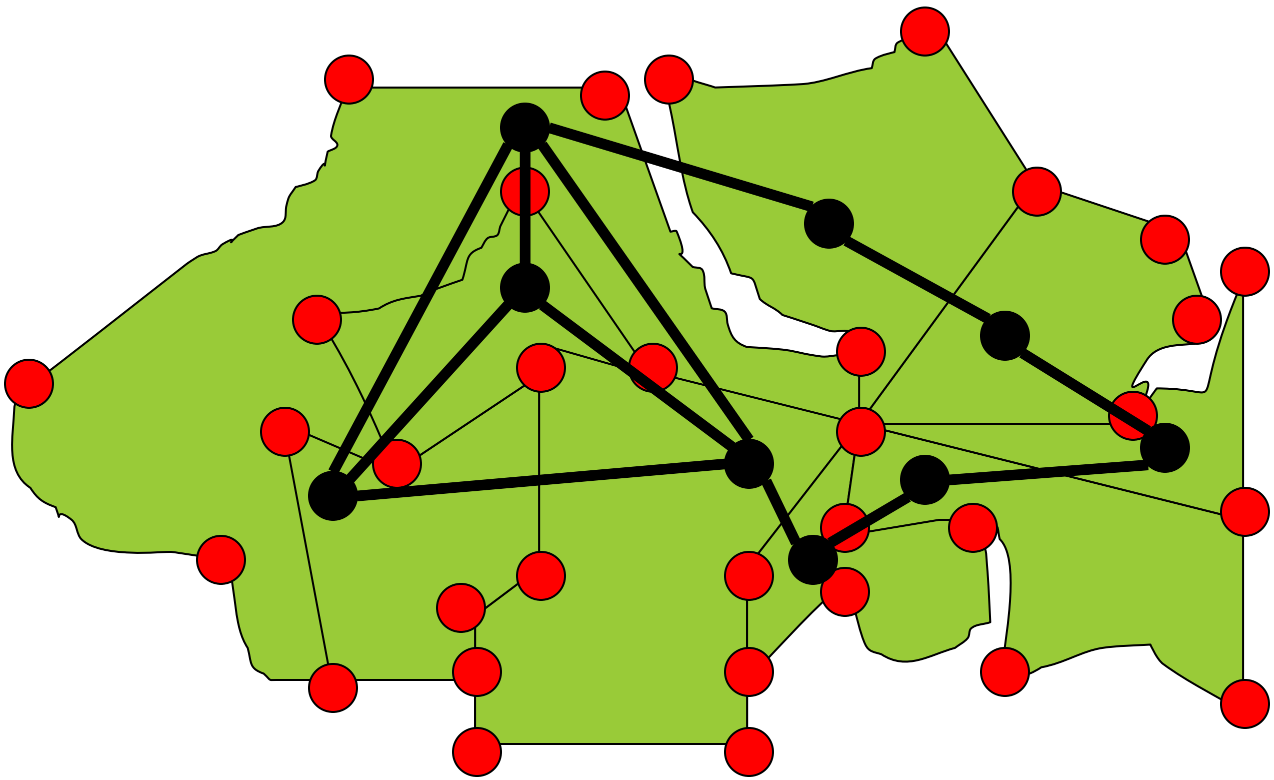
1) Put vertex inside each region:



From Maps to Graphs to Dual Graphs

Dual Graphs :

2) Connect vertices across common edges:



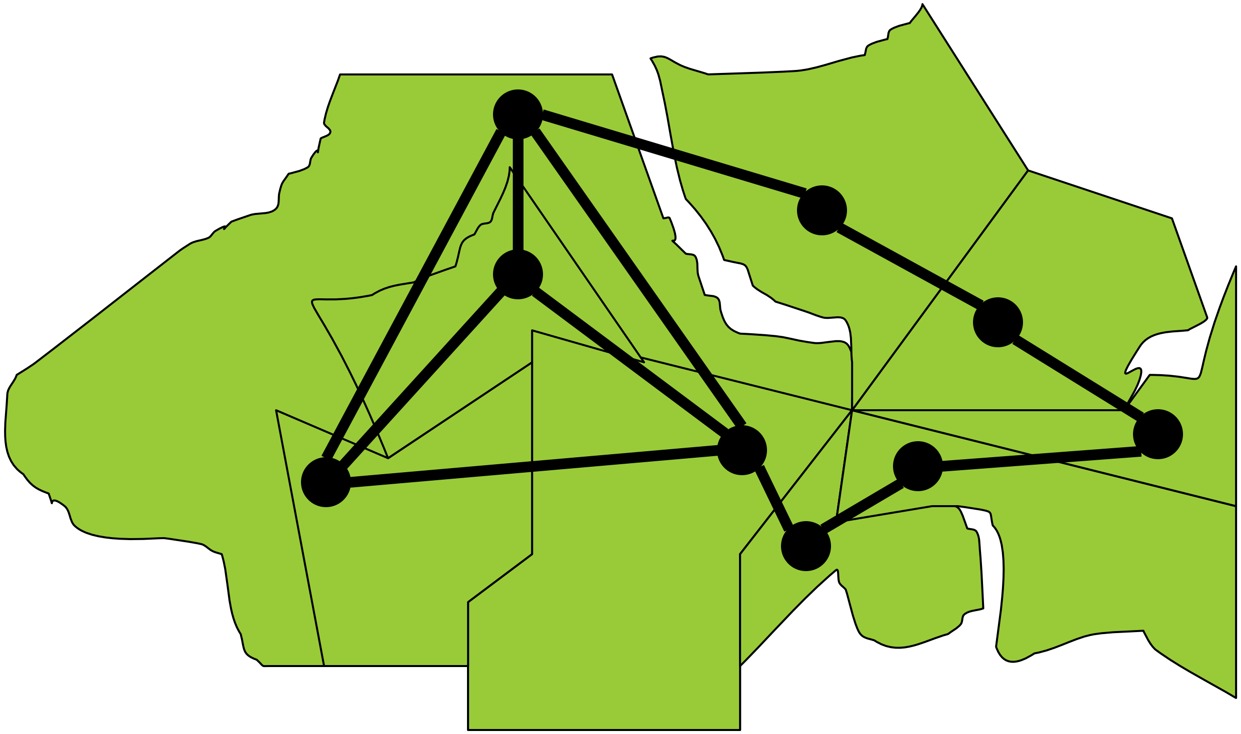
Definition of Dual Graph

DEF: The ***dual graph*** $G^\wedge$ of a planar graph $G = (V, E, R)$ [Vertices, Edges, Regions] is the graph obtained by setting

- Vertices of $G^\wedge$: $V(G^\wedge) = R$
- Edges of $G^\wedge$: $E(G^\wedge)$ = set of edges of the form $\{F_1,F_2\}$ where F_1 and F_2 share a common edge.

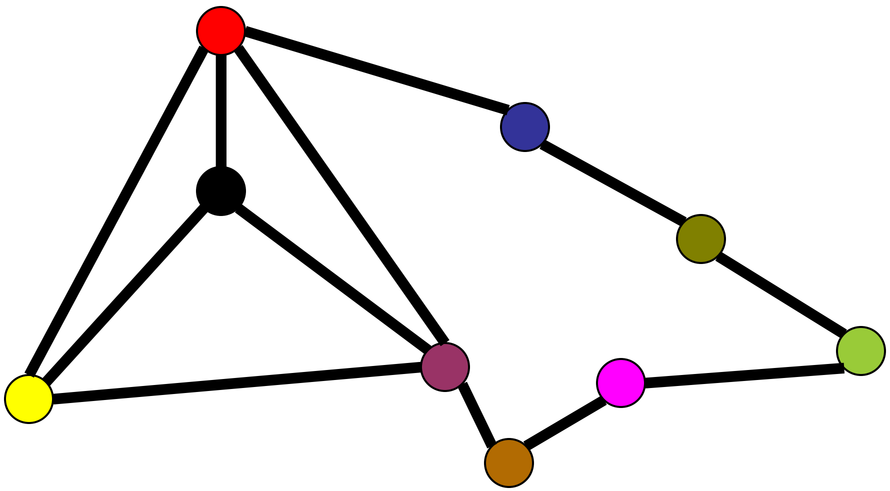
From Maps to Graphs to Dual Graphs

So take dual graph:



From Map Coloring to Graph Coloring

Coloring regions is equivalent to coloring vertices of dual graph.



Definition of Colorable

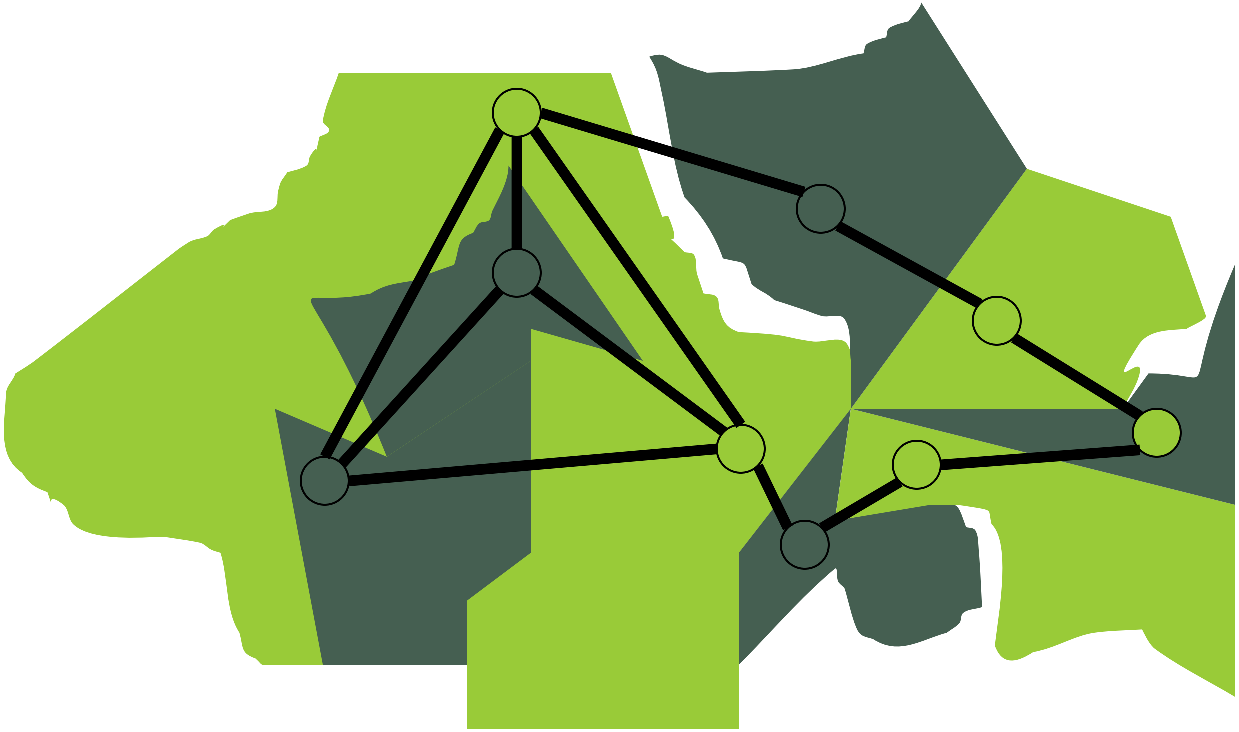
DEF: Let *n* be a positive number. A simple graph is *n* -***colorable*** if the vertices can be colored using *n* colors so that no two adjacent vertices have the same color.

The ***chromatic number*** of a graph is smallest number *n* for which it is *n* -colorable.

EG: A graph is bipartite iff it is 2-colorable.

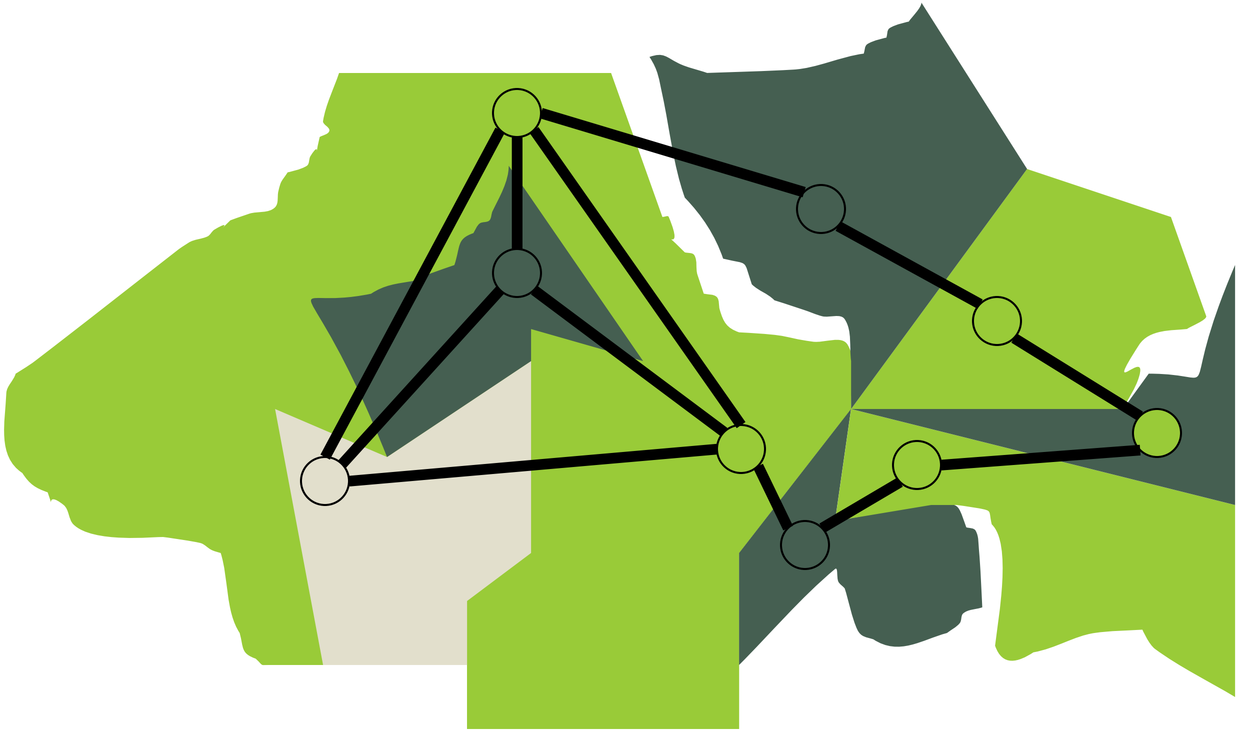
From Map Coloring to Graph Coloring

Map not 2-colorable, so dual graph not 2-colorable:



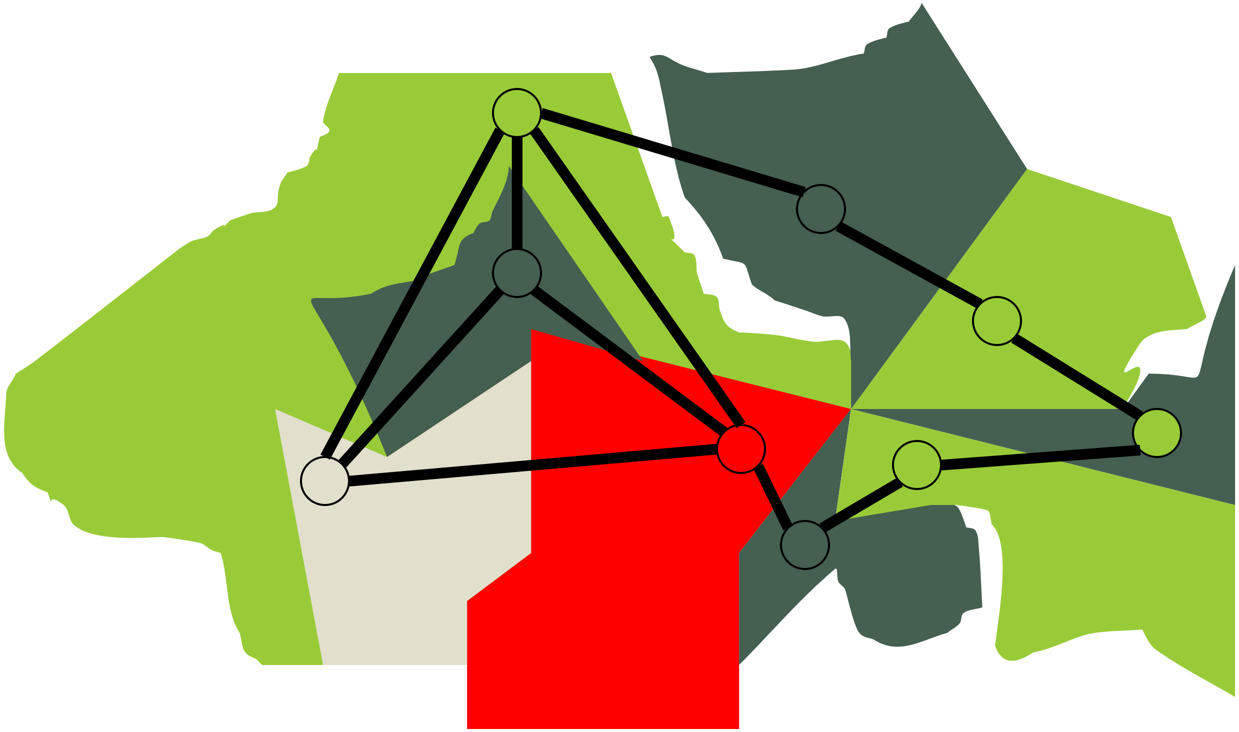
From Map Coloring to Graph Coloring

Map not 3-colorable, so graph not 3-colorable:



From Map Coloring to Graph Coloring

Graph is 4-colorable, so map is as well:



4-Color Theorem –Graph Theory Version

THM: Any planar graph is 4-colorable.

Graph Coloring and Schedules

EG: Suppose want to schedule some final exams for CS courses with following call numbers:

1007, 3137, 3157, 3203, 3261, 4115, 4118, 4156

Suppose also that there are no common students in the following pairs of courses because of prerequisites:

1007-3137

1007-3157, 3137-3157

1007-3203

1007-3261, 3137-3261, 3203-3261

1007-4115, 3137-4115, 3203-4115, 3261-4115

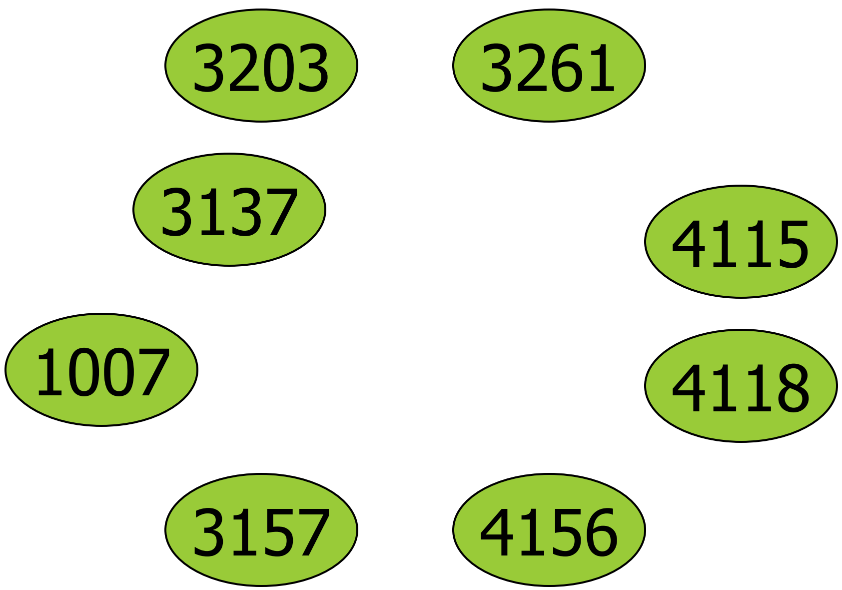
1007-4118, 3137-4118

1007-4156, 3137-4156, 3157-4156

How many exam slots are necessary to schedule exams?

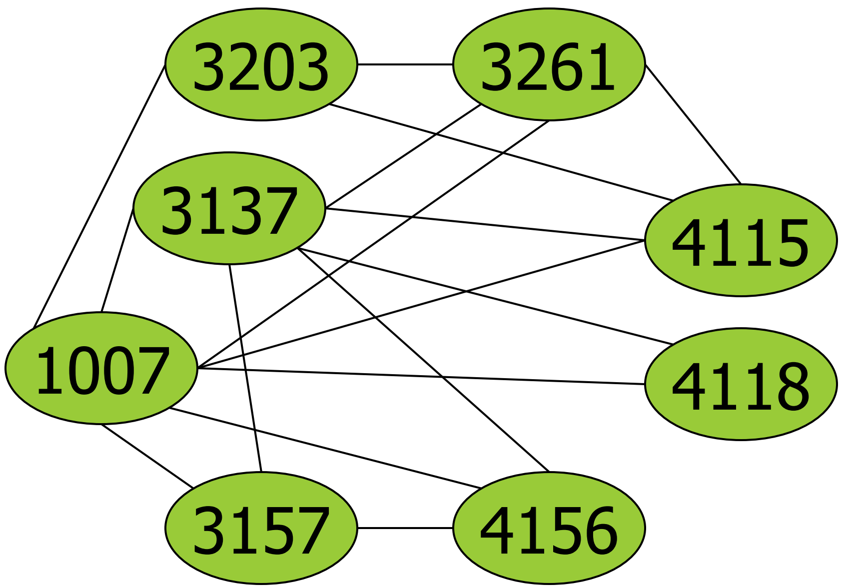
Graph Coloring and Schedules

Turn this into a graph coloring problem. Vertices are courses, and edges are courses which *cannot* be scheduled simultaneously because of possible students in common:



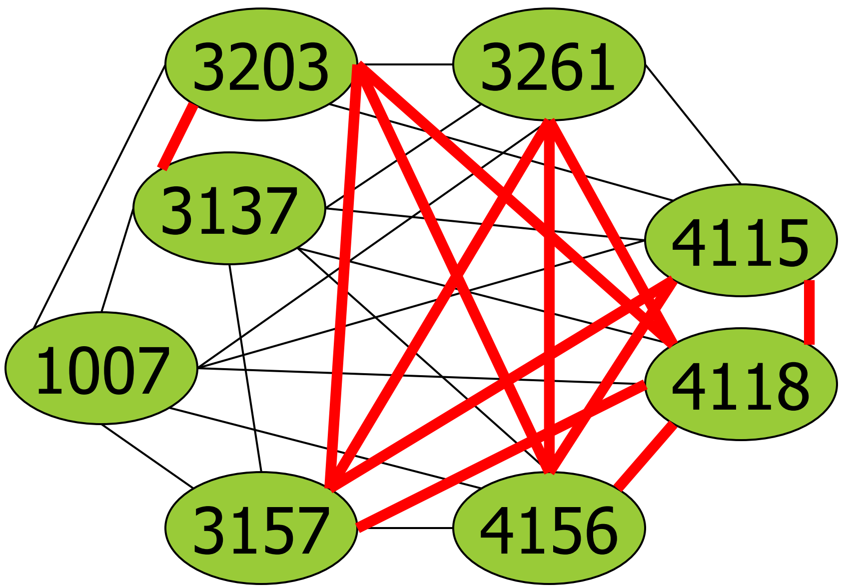
Graph Coloring and Schedules

One way to do this is to put edges down where students mutually excluded…



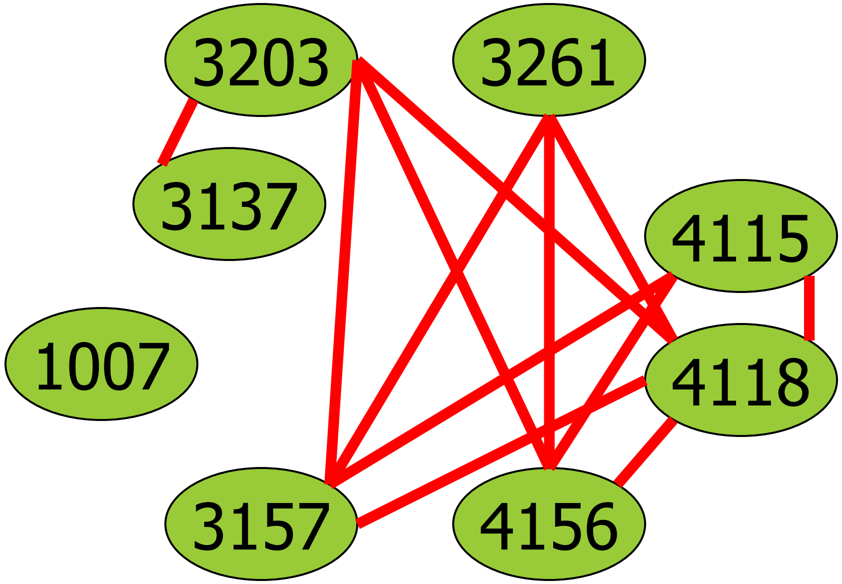
Graph Coloring and Schedules

…and then compute the complementary graph:



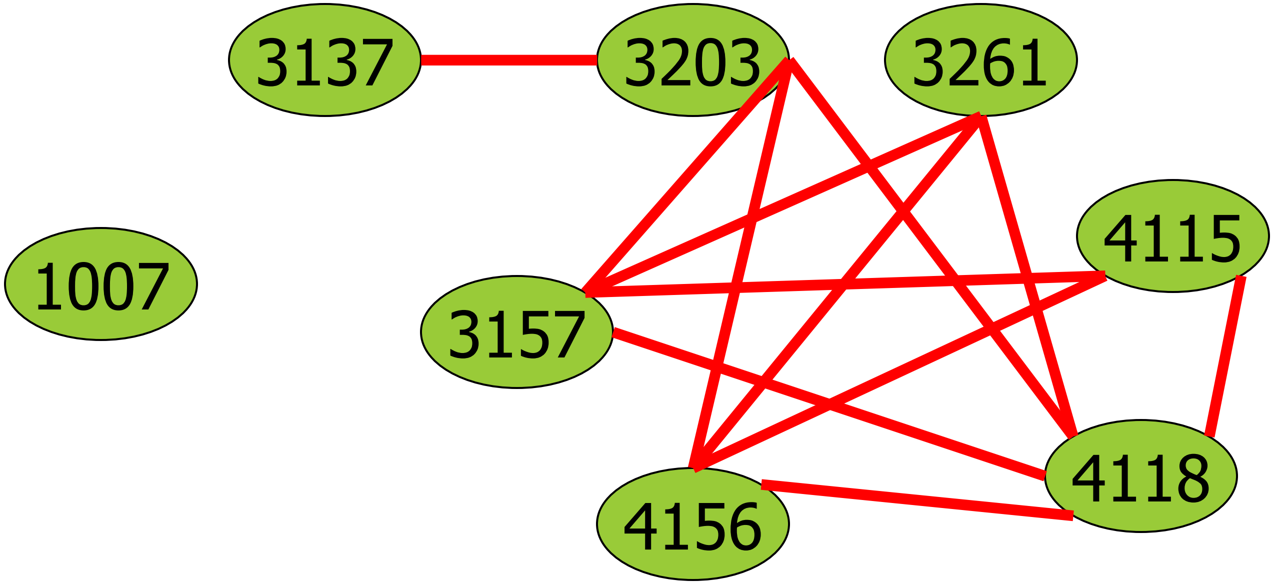
Graph Coloring and Schedules

…and then compute the complementary graph:



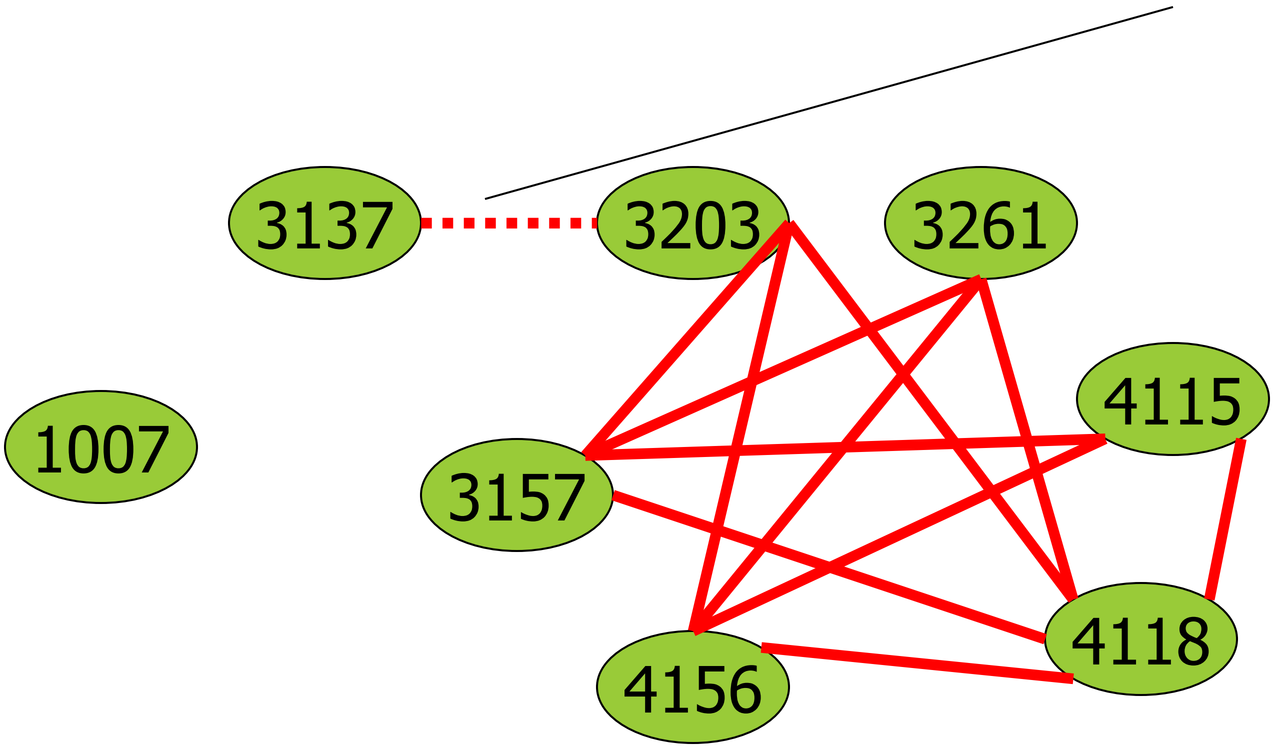
Graph Coloring and Schedules

Redraw:



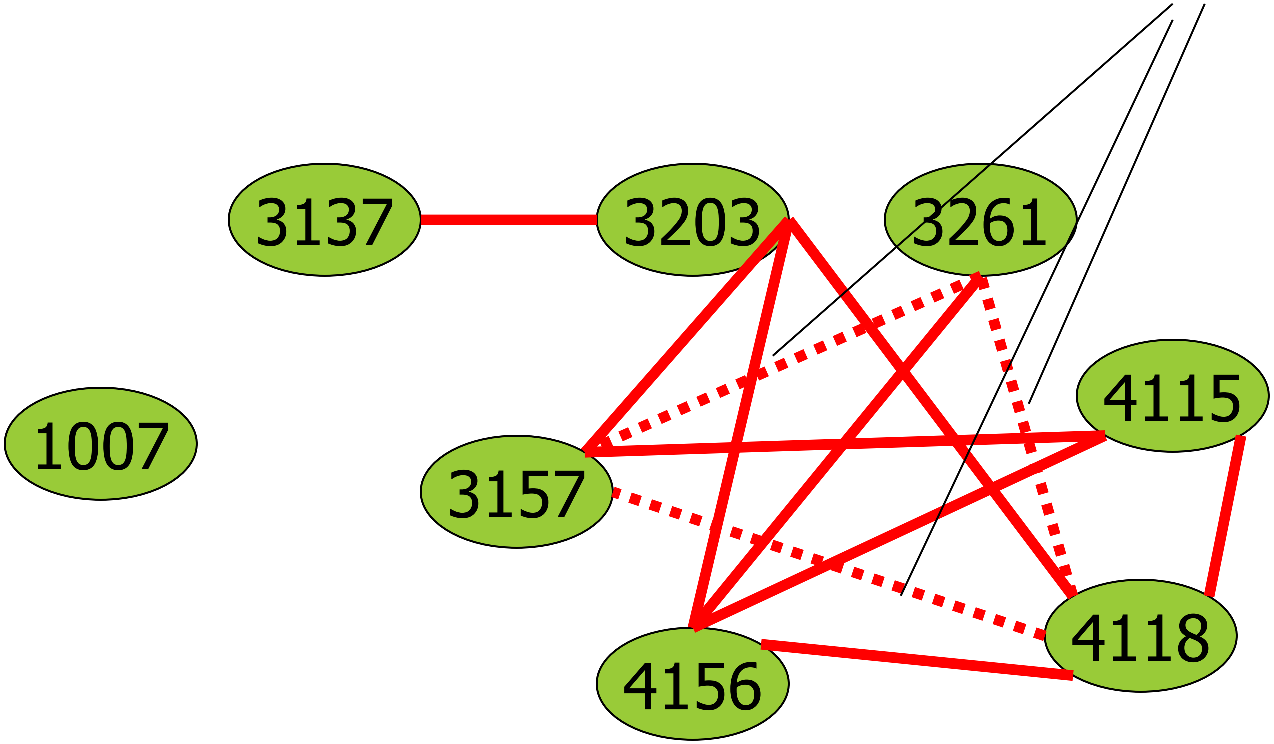
Graph Coloring and Schedules

Not 1-colorable because of edge



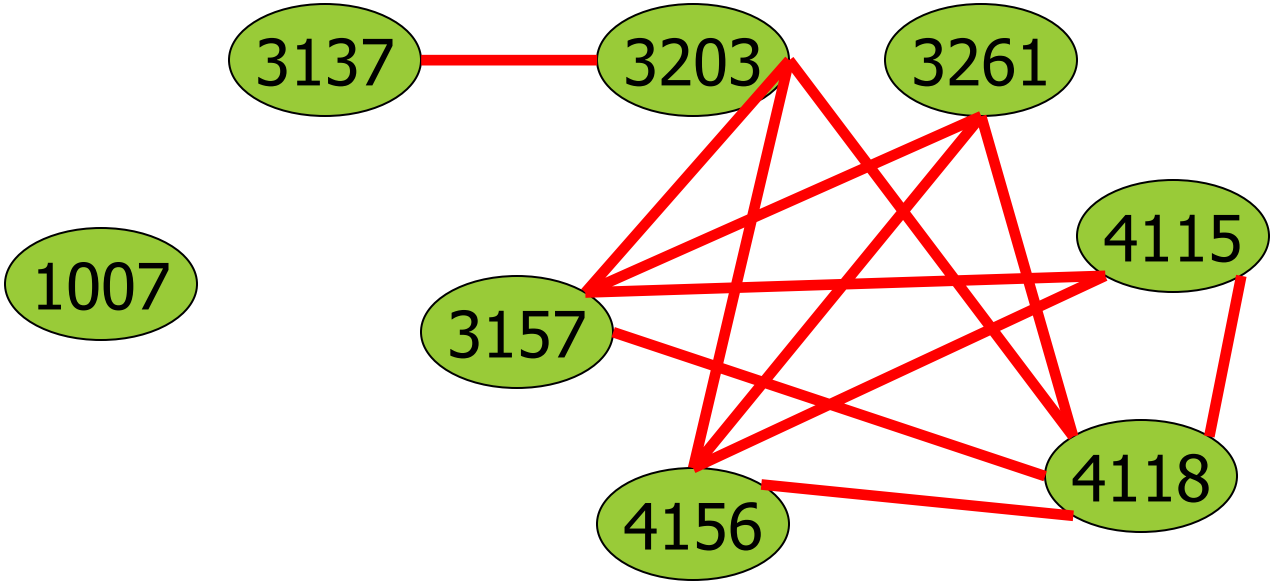
Graph Coloring and Schedules

Not 2-colorable because of triangle



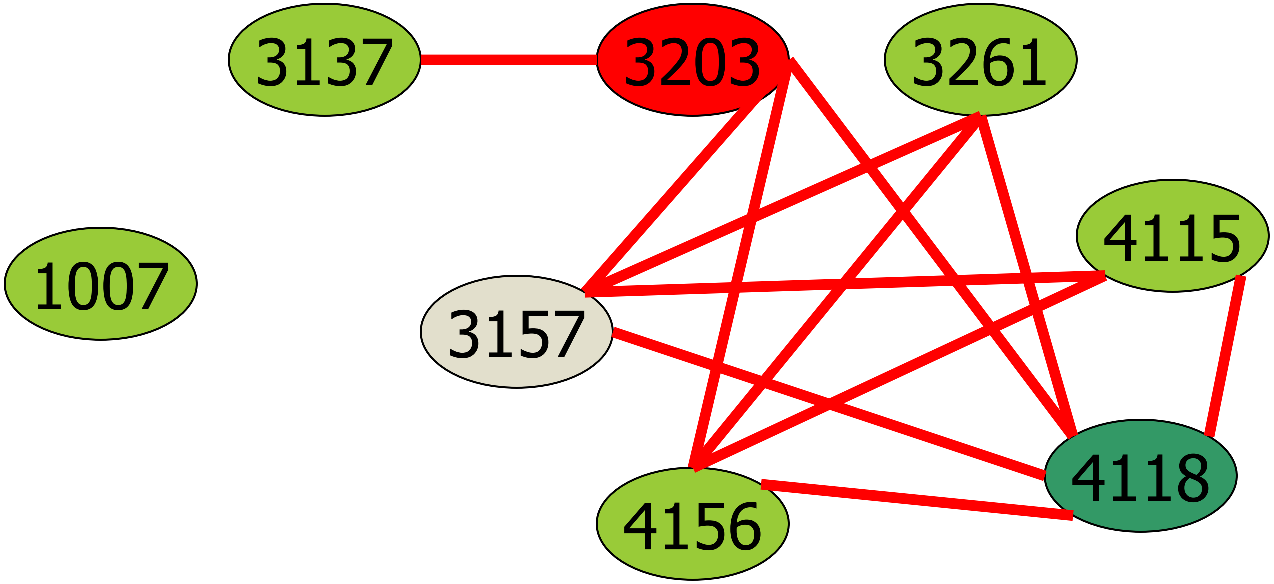
Graph Coloring and Schedules

Is 3-colorable. Try to color by Red, Green, Blue.



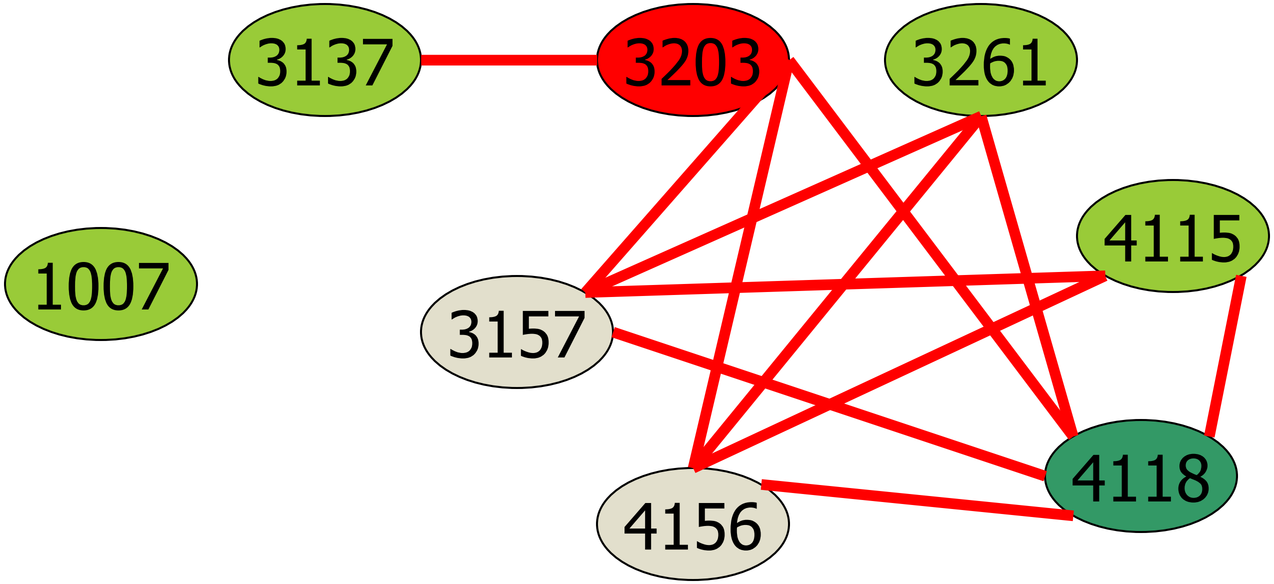
Graph Coloring and Schedules

WLOG. 3203-Red, 3157-Blue, 4118-Green:



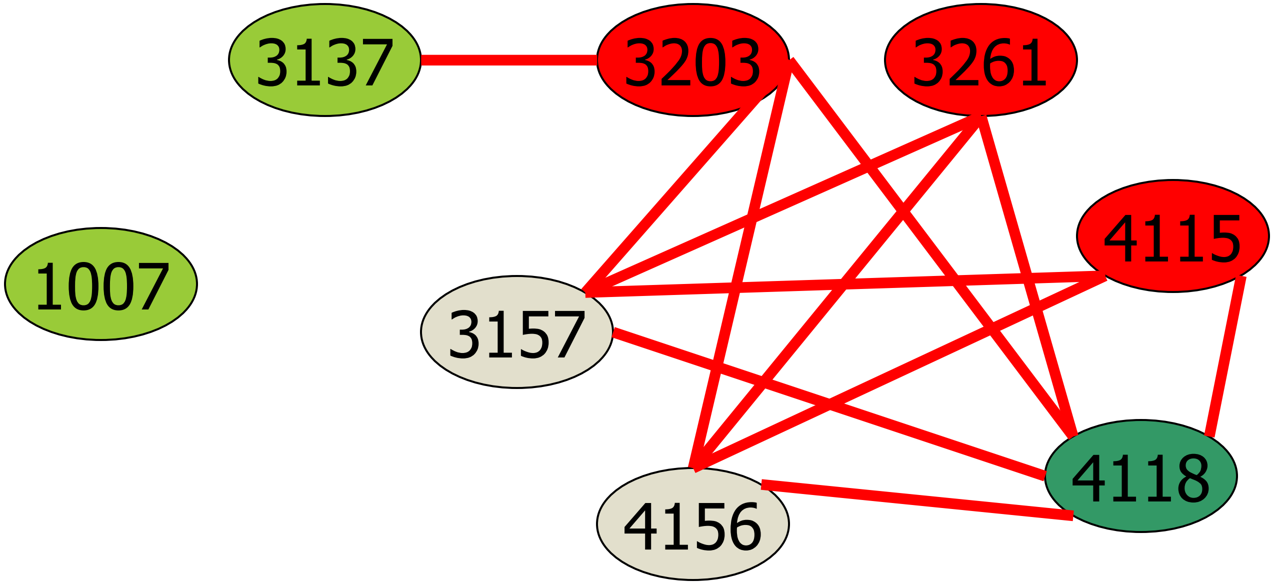
Graph Coloring and Schedules

So 4156 must be Blue:



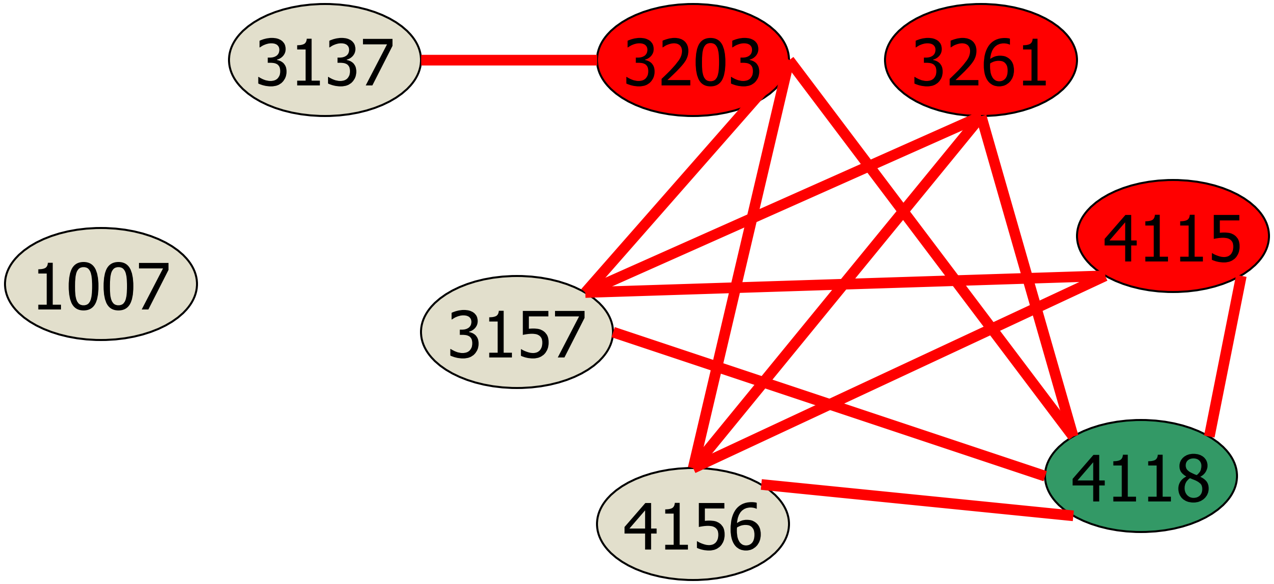
Graph Coloring and Schedules

So 3261 and 4115 must be Red.



Graph Coloring and Schedules

3137 and 1007 easy to color.



Graph Coloring and Schedules

So need 3 exam slots: