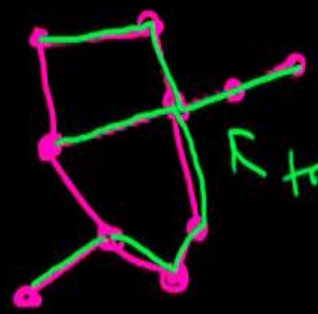
Theorem: A graph G is a tree $\Leftrightarrow$ G is acyclic and $|E(G)| = |V(G)| - 1$

$$m = n - 1$$

Corollary: A forest F of order n with k connected components has $m = n - k$ edges.

spanning tree of a graph G: $\underbrace{\text{spanning}}_{\text{touches all of the vertices of } G}$ subgraph of G that is a tree



tree

Every connected graph G has a spanning tree

- if G is a tree ✓
- if G is not a tree

G

remove edges of cycles until only bridges remain

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- Every connected graph G has a spanning tree
  - if G is a tree ✓
  - if G is not a tree

$$m - \underbrace{(m - (n-1))}_{\text{\# of edges that need removing}} = \underbrace{n - 1}_{\text{tree}}$$

remove edges of cycles until only bridges remain

Corollary: Every connected graph has $m \geq n-1$

Theorem: A graph G is a tree $\iff$ G is connected and has $m = n-1$

Proof: $\Rightarrow]$ G is a tree $\Rightarrow$ connected ✓

$\hookrightarrow$ prev. Thm $\Rightarrow m = n-1$ ✓

$\Leftarrow]$ G is connected $m = n-1$

Suppose C is a cycle in G

$e \in C$

$G \setminus e$ $\leftarrow$ still connected

$G \setminus e$ has $n-2$ $(\Rightarrow\Leftarrow)$

G is acyclic $\therefore$ a tree :) $\square$

Trees: connected – –

acyclic – –

$m = n - 1$ – –

you have a tree :)

you have a tree :)

you have a tree :)

Types of Trees

P_5

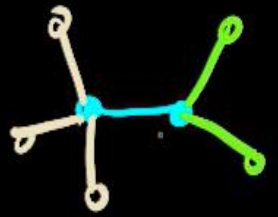
Star

$K_{1,5}$

↑

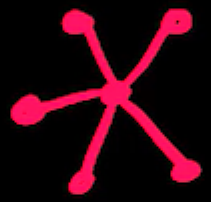
only 1

double star



P_5

star



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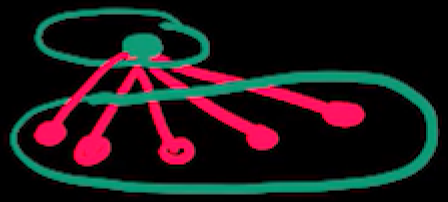

$K_{1,5}$

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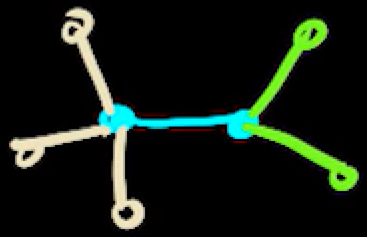
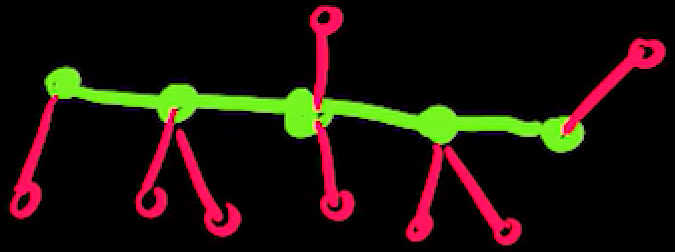
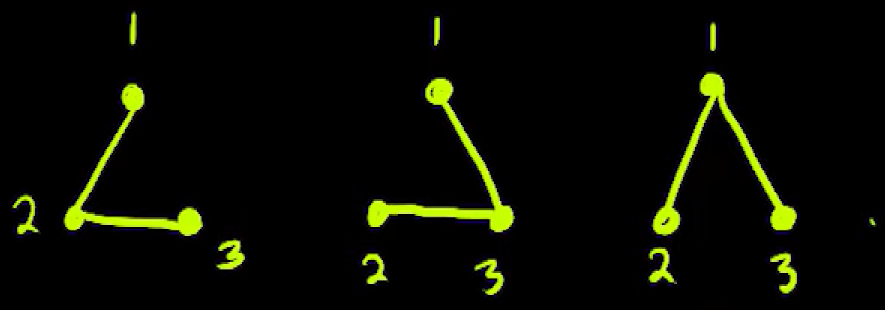




Cayley's Formula: There are n^{n-2} labelled trees of order n.



Proof Idea: Labelled tree T $\longleftrightarrow$ Prüfer sequence S

1:1 correspondence

on n elements of length $n-2$

How many such seq?

$n \times n \times n \times \cdots \times n = n^{n-2}$

$$\overbrace{\underbrace{\underline{1}\;\underline{n}\;\underline{4}\;\underline{4}\;\cdots\;\underline{}}_{n-2}}$$

Proof Idea: Labelled tree $T \xleftrightarrow{\text{1:1 correspondence}}$ Prüfer sequence S on n elements of length $n-2$

$$\underbrace{\overbrace{\underline{1}\ \underline{n}\ \underline{4}\ \underline{4}\ \cdots\ \underline{\ \ }}}_{n-2}$$

How many such seq?

$n \times n \times n \times \cdots \times n = n^{n-2}$

labelled

$T \to$ Prüfer sequence S

- Find a leaf of T with smallest label
  - $\to$ add the neighbour to S
  - $\to$ delete this leaf
- Repeat until our tree is K_2 $\bullet\!\!-\!\!-\!\!\bullet$

labelled

n-2

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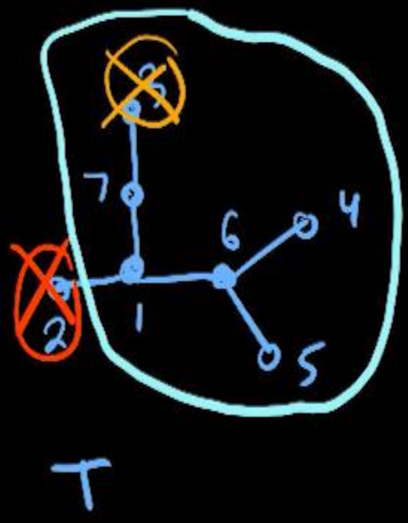

$S = (\ )$

$S = (1)$

$S = (1, 7)$

labelled $n-2$

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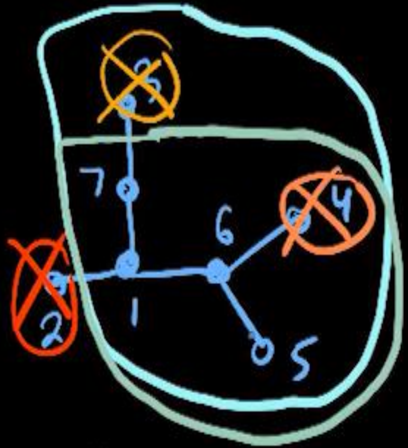

T

$S = (\ )$

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$S = (1, 7)$

$S = (1, 7, 6)$

labelled

$n-2$

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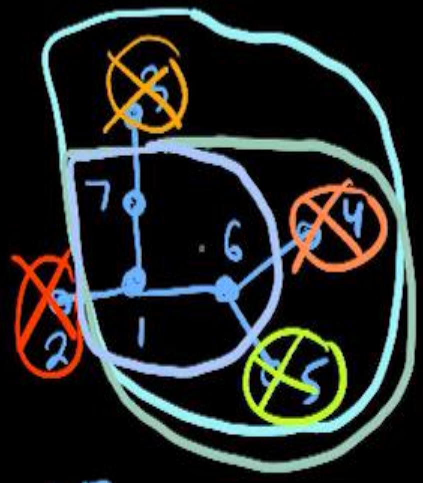

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$S = (1, 7, 6, 6)$

labelled

$n-2$

Press Esc to exit full screen

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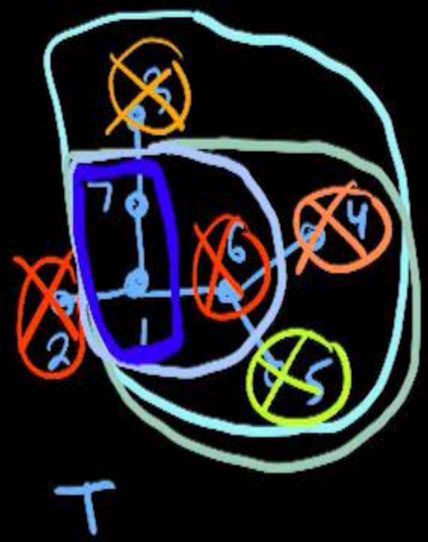


$S = (\ )$

$S = (1)$

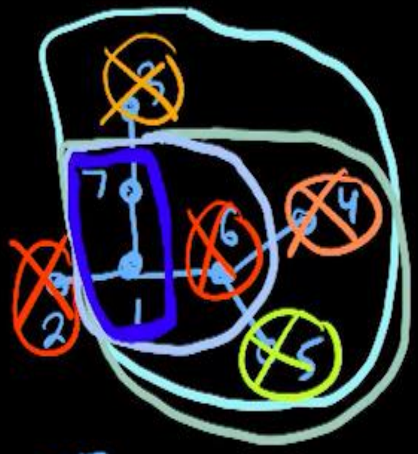
$S = (1, 7)$

$S = (1, 7, 6)$

$S = (1, 7, 6, 6)$

$\boxed{S = (1, 7, 6, 6, 1)}$

• Repeat until our tree is K_2



T

$n = 7$

$S = (\ )$

$S = (1)$

$S = (1, 7)$

$S = (1, 7, 6)$

$S = (1, 7, 6, 6)$

$$\underbrace{\boxed{S = (1, 7, 6, 6, 1)}}_{n-2}$$

- no leaf gets appended to S

- every vertex v is added to S a total of $\deg(v) - 1$ times

T $\quad n=7 \quad$ $S = (1, 7, 6)$

- no leaf gets appended to S
- every vertex v is added to S a total of $\deg(v)-1$ times

n vertices, m edges $\Rightarrow m = n-1$

$$\#\text{ of terms in } S = \sum_{v \in V(T)} (\deg(v) - 1) = \underbrace{\left(\sum_{v \in V(T)} \deg v\right)}_{2m} - \underbrace{\sum_{v \in V(T)} 1}_{n}$$

$$= 2m - n$$

$$= 2(n-1) - n$$

$$= n - 2$$

Prüfer sequence S of length $n-2$ using symbols $1,\dots,n$ $\longrightarrow$ labelled tree T

- $S = (a_1, a_2, \dots, a_{n-2})$ $\quad a_i \in \underbrace{\{1,\dots,n\}}_{L}$
- Find smallest element $x \in \{1,\dots,n\}$ $\quad x \notin S$

↳ join x to 1st element of S

→ delete a_1 from S, delete x from $\underbrace{\{1,\dots,n\}}_{L}$

- Find smallest $y \in L \setminus \{x\}$ not in S

- $S=(a_1,a_2,\dots,a_{n-2})$ $\quad a_i \in \underbrace{\{1,\dots,n\}}_{L}$
- Find smallest element $x\in\{1,\dots,n\}$ $\; x\notin S$

  ↳ join x to 1st element of S

  → delete a_1 from S, delete x from $\underbrace{\{1,\dots,n\}}_{L}$

- Find smallest $y\in L\setminus\{x\}$ not in $S\setminus\{a_1\}$

  ↳ join y to the 1st element of $S\setminus\{a_1\}$

  → delete a_2 from S

  delete y from L

↳ join x to 1^{st} element of S

→ delete a_1 from S, delete x from $\underbrace{\{1, \dots, n\}}_{L}$

- Find smallest $y \in L \setminus \{x\}$ not in $S \setminus \{a_1\}$

  ↳ join y to the 1^{st} element of $S \setminus \{a_1\}$

  → delete a_2 from S
  delete y from L

- Continue until 2 items remain in L

  ↳ join these two via an edge

delete y from L $\qquad y \quad a_2$

- Continue until 2 items remain in L
  ↳ join these two via an edge

$S = (1, 7, 6, 6, 1)$ $\quad$ length 5 $\Rightarrow$ $\boxed{n = 7}$

$L = \{1, 2, \ldots, 7\}$

$x \in L \setminus S$ smallest $\to x = 2$

delete y from L $\qquad$ y $\quad a_2$

- Continue until 2 items remain in L
  ↳ join these two via an edge

$S = (\cancel{1}, 7, 6, 6, 1)$ $\quad$ length 5 $\Rightarrow \boxed{n = 7}$

$L = \{1, \cancel{2}, \ldots, 7\}$

$x \in L \setminus S$ smallest $\rightarrow x = 2$

$S = (7, 6, 6, 1)$

$L = \{1, 3, 4, 5, 6, 7\}$

delete y from L $\qquad y \quad a_2$

- Continue until 2 items remain in L
  ↳ join these two via an edge

$S = (\cancel{1}, 7, 6, 6, 1)$ length 5 $\Rightarrow$ $\boxed{n = 7}$

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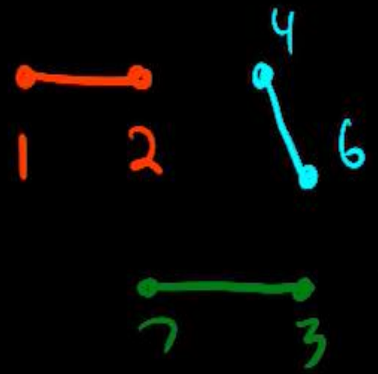
$S = (\cancel{7}, \cancel{6}, 6, 1)$

$L = \{1, \cancel{3}, \cancel{4}, 5, 6, 7\}$



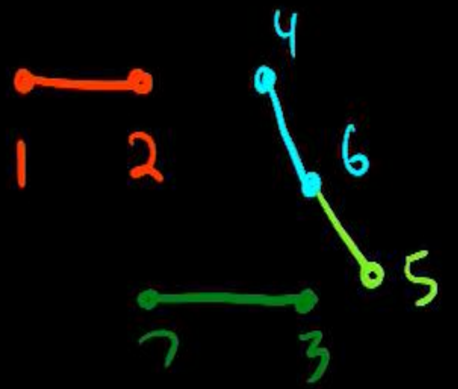
delete y from L $\qquad\qquad$ $y \quad a_2$

- Continue until 2 items remain in L
  ↳ join these two via an edge

$S = (\cancel{4}, 7, 6, 6, 1)$ $\quad$ length 5 $\Rightarrow$ $\boxed{n=7}$

$L = \{1, \cancel{2}, \ldots, 7\}$

$x \in L \setminus S$ smallest $\rightarrow x = 2$



$S = (\cancel{7}, \cancel{6}, \cancel{6}, 1)$

$L = \{1, \cancel{3}, \cancel{4}, \cancel{5}, 6, 7\}$

delete y from L $\qquad\qquad$ y a_2

- Continue until 2 items remain in L
  - ↳ join these two via an edge

$S = (\cancel{1}, 7, 6, 6, 1)$ $\quad$ $\underline{\text{length } 5} \Rightarrow \boxed{n = 7}$

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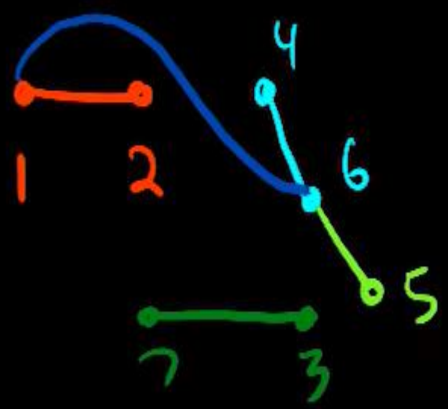

$S = (\cancel{7}, \cancel{6}, \cancel{6}, \cancel{1})$

$L = \{1, \cancel{3}, \cancel{4}, \cancel{5}, \cancel{6}, 7\}$

delete y from L y a_2

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  ↳ join these two via an edge

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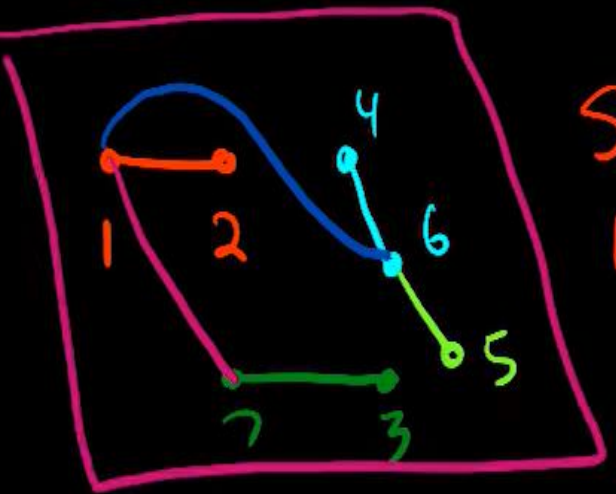
$L = \{1, \cancel{3}, \cancel{4}, \cancel{5}, \cancel{6}, 7\}$

T



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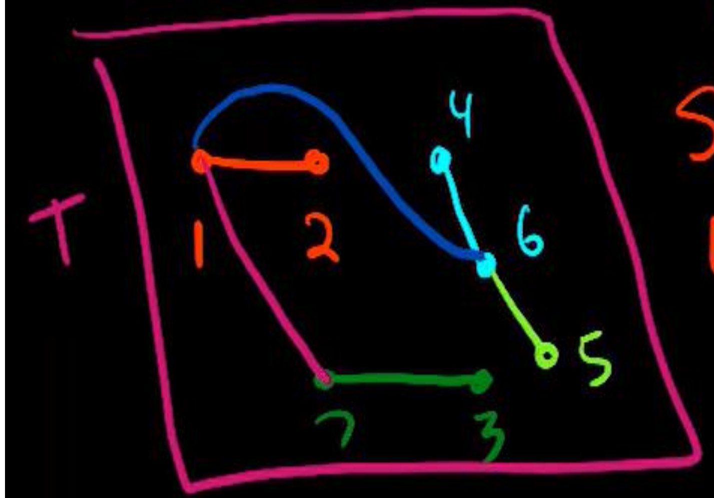

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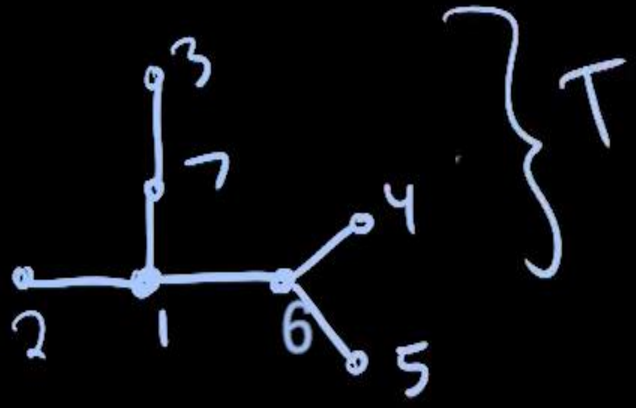
$L = \{1, \cancel{3}, \cancel{4}, \cancel{5}, \cancel{6}, 7\}$



$S = (1, 7, 6, 6, 1)$ length $5 \Rightarrow n = 7$

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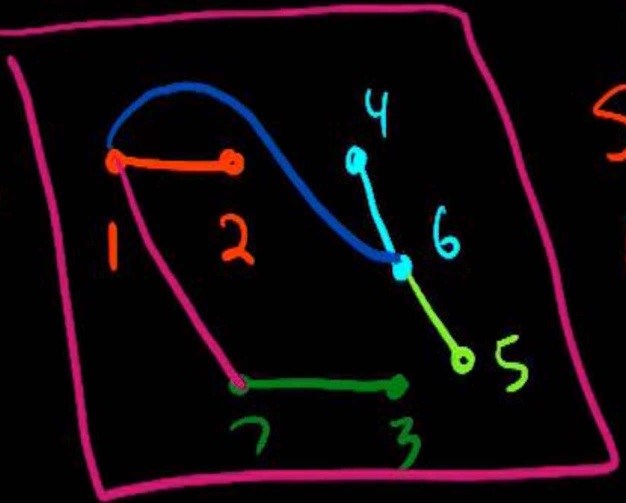
T



$S = (\cancel{7}, \cancel{6}, \cancel{6}, \cancel{1})$

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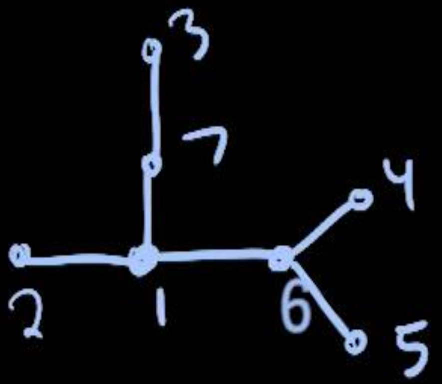
$S \to T$



$\}\, T$

$T \to S$

Thank you