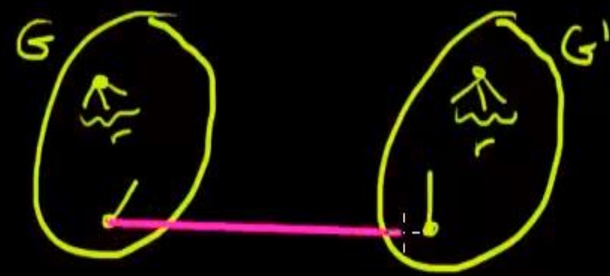
# Subgraphs of regular graphs

Theorem (König): Every graph $G$ with $\Delta(G) = r$ is an induced subgraph of an $r$-regular graph.

Proof: If $G$ is $r$-regular, then we're done.

Suppose $G$ is not $r$-regular, ie $\delta(G) < r$

Let $G'$ be another copy of $G$ and join corresponding vertices of $G$ and $G'$ with an edge if they had degree $< r$

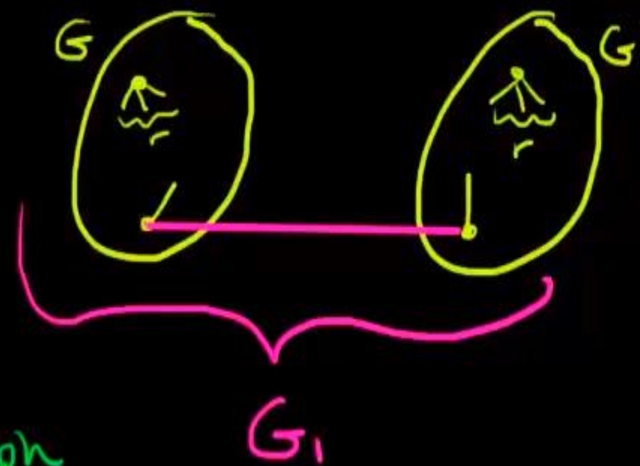


**Proof:** If $G$ is $r$-regular, then we're done.

Suppose $G$ is not $r$-regular, ie $\delta(G) < r$

Let $G'$ be another copy of $G$ and join corresponding vertices of $G$ and $G'$ with an edge if they had degree $< r$

Call the resulting graph $G_1$




If $G_1$ is $r$-regular, then we're done since $G$ is an induced subgraph of $G_1$.

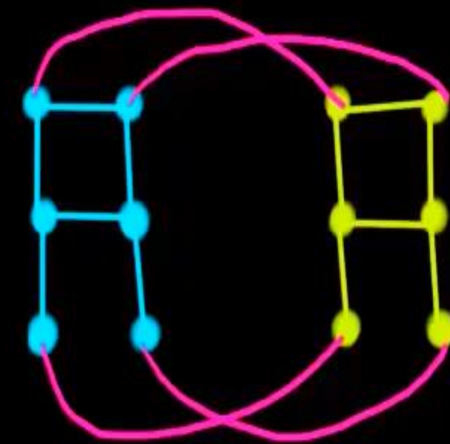
If not, then continue the procedure until arriving at an $r$-regular graph $G_K$ where $K = r - \delta(G)$

Ex: G

$\Delta(G) = 3$

$\delta(G) = 1$

$\longrightarrow$

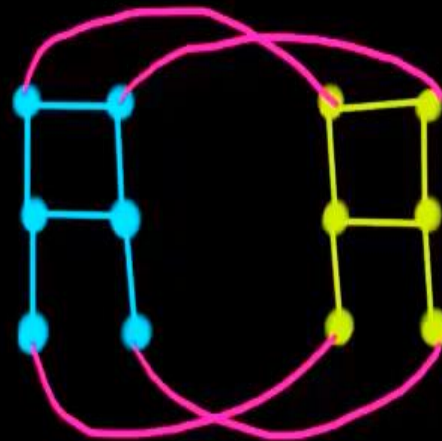


$G_1$

Ex: $G$

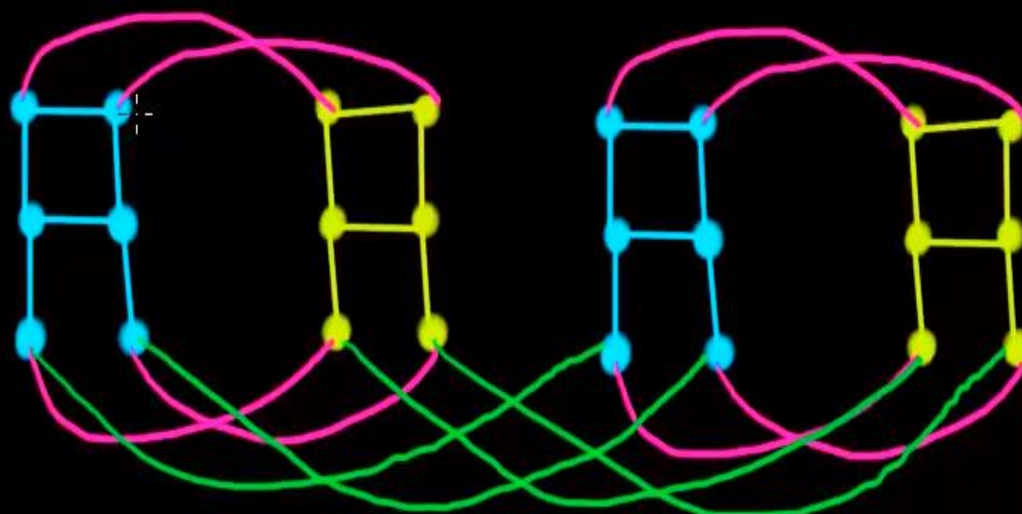
$\Delta(G) = 3$

$\delta(G) = 1$



$G_1$

$G_2$ is 3-regular

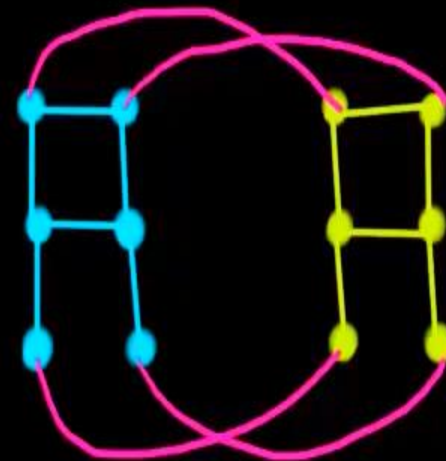


$G_2$

Ex: $G$

$\Delta(G) = 3$

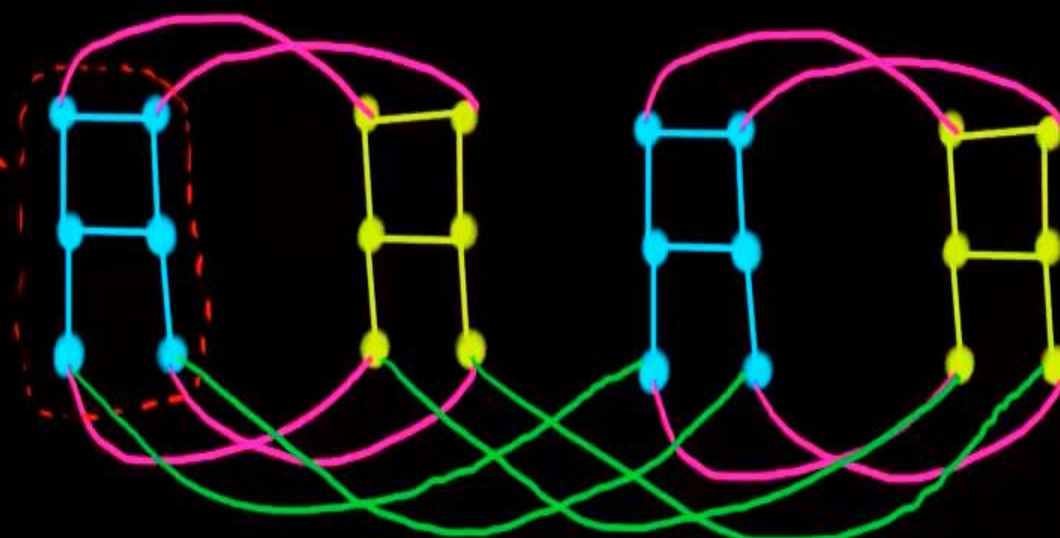
$\delta(G) = 1$



$G_1$

$G_2$ is 3-regular

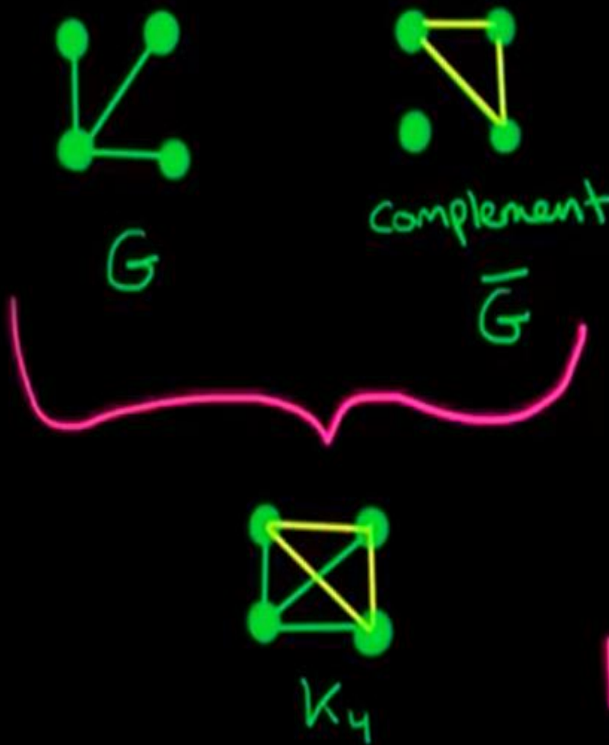
the induced subgraph on these vertices is $G$



$G_2$

# Complement of a graph

Ex:




Let $G = (V, E)$ be a graph of order $n$.
The complement graph $\overline{G}$ is the graph with

$$V(\overline{G}) = V(G)$$

and

$$E(\overline{G}) = E(K_n) \setminus E(G)$$

$$|E(\overline{G})| = \binom{n}{2} - m$$

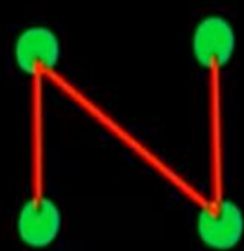
where $m = |E(G)|$

$\binom{4}{2} = 6$ $K_4$

$$|E(\overline{G})| = \binom{n}{2} - m$$

where $m = |E(G)|$

A graph $G$ is called self-complementary if $G \cong \overline{G}$

$G$   $\overline{G}$

$P_4$ $P_4$

$G$   $\overline{G}$

$G \cong \overline{G} \cong C_5$

$\binom{4}{2} = 6$

$K_4$

$$|E(\overline{G})| = \binom{n}{2} - m$$

where $m = |E(G)|$

A graph $G$ is called self-complementary if $G \cong \overline{G}$

$G$

$P_4$

$P_4$

$\overline{G}$

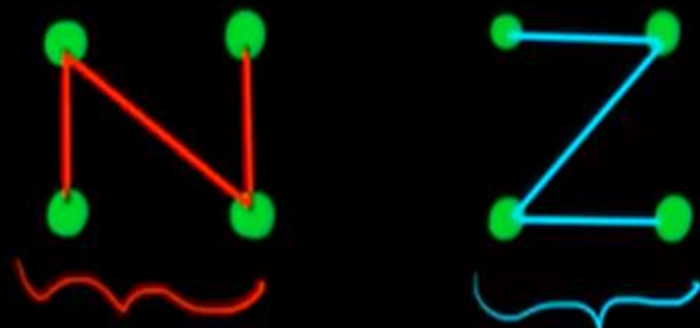
$\binom{4}{2} = 6$ and $G, \overline{G}$ each have 3 edges

$G$

$\overline{G}$

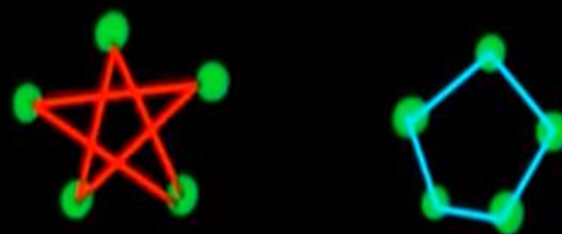
$\binom{5}{2} = 10$ and $G, \overline{G}$ each have 5 edges

$G \cong \overline{G} \cong C_5$

$G$  $\bar{G}$

$P_4$ $\quad$ $P_4$

$\binom{4}{2} = 6$ and $G, \bar{G}$ each have 3 edges

$G$  $\bar{G}$

$\binom{5}{2} = 10$ and $G, \bar{G}$ each have 5 edges

$G \cong \bar{G} \cong C_5$

Question: Is there a self-complementary graph of order 6?

$\hookrightarrow$ NO since $\binom{6}{2} = 15$ is <u>odd</u>

Question: Is there a self-complementary graph of order 6?

↳ NO since $\binom{6}{2} = 15$ is <u>odd</u>

Fact: A self-complementary graph of order $n$ must have

$$\frac{1}{2}\binom{n}{2} = \frac{n(n-1)}{4} \text{ edges}$$

$\therefore$ If $G$ of order $n$ is self-complementary

then $n(n-1)$ must be divisible by 4

so $n \equiv 0$ or $1 \pmod 4$.

$n = 4$ ✓

$n = 5$ ✓

$n = 6$ ✗

$\therefore$ If $G$ of order $n$ is self-complementary then $n(n-1)$ must be divisible by 4

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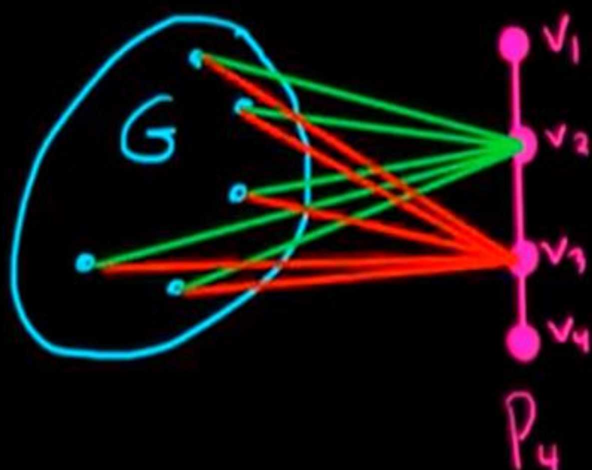
$n=4$ ✓
$n=5$ ✓
$n=6$ ✗
$n=7$ ✗
$n=8$
$n=9$

Given a self-complementary graph $G$ of order $n$ we can construct a self-complementary graph $G'$ of order $n+4$ as follows:




Add $P_4 = [v_1, v_2, v_3, v_4]$ and join vertex $v_2$ and vertex $v_3$ to all vertices of $G$

$\therefore$ If $G$ of order $n$ is self-complementary then $n(n-1)$ must be divisible by 4

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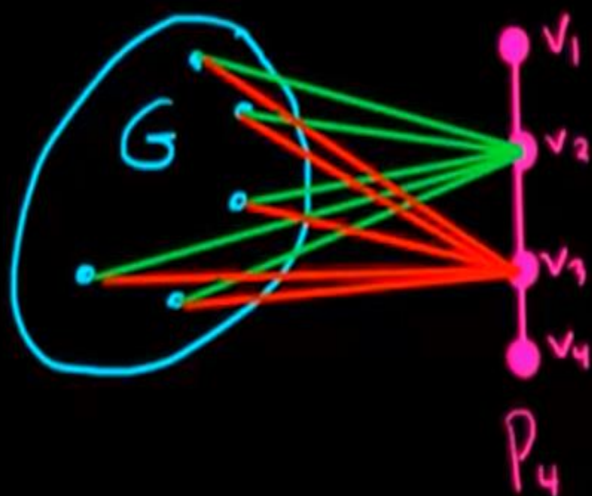
$n=4$ ✓ $n=8$ ✓ $n=12$ ✓

$n=5$ ✓ $n=9$ ✓ $n=13$ ✓

$n=6$ ✗

$n=7$ ✗

Given a self-complementary graph $G$ of order $n$ we can construct a self-complementary graph $G'$ of order $n+4$ as follows:

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# Cartesian Product of Graphs

Recall: The cartesian product of 2 sets A and B is

$$A \times B = \{ (a,b) \mid a \in A,\ b \in B \}$$

Ex: $A = \{x, y\}$ $B = \{1, 2, 3\}$

Then $A \times B = \{ (x,1), (x,2), (x,3), (y,1), (y,2), (y,3) \}$

Ex: Euclidean plane : $\mathbb{R} \times \mathbb{R}$

Note: $\mathbb{R}$ is the set of real numbers


$x \in \mathbb{R}$
$y \in \mathbb{R}$

Suppose $G$ and $H$ are graphs with $V(G) = \{u_1, u_2, \dots, u_m\}$
and $V(H) = \{v_1, v_2, \dots, v_n\}$

can have different # of vertices

Then $G \times H$ is the graph with vertex set

$$V(G \times H) = V(G) \times V(H)$$
$$= \{ (u_i, v_j) \mid u_i \in V(G),\ v_j \in V(H) \}$$

and $e$ is an edge of $G \times H$ <u>iff</u> $e = (u_i, v_j)(u_k, v_\ell)$

where either:

1) $i = k$ and $v_j v_\ell \in E(H)$

or 2) $j = \ell$ and $u_i u_k \in E(G)$

Ex: $K_2 \times P_4$

$K_2$ 


$P_4$ 

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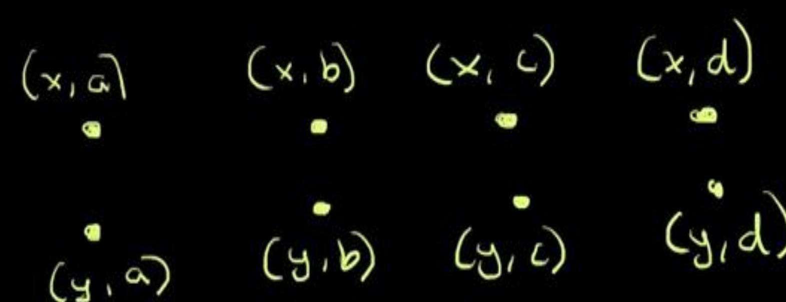

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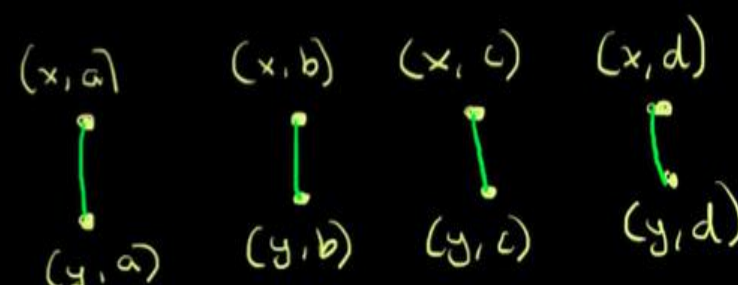


$= \{ (u_i, v_j) \mid u_i \in V(G),\ v_j \in V(H) \}$

and $e$ is an edge of $G \times H$ **iff** $e = (u_i, v_j)(u_k, v_l)$

where either:

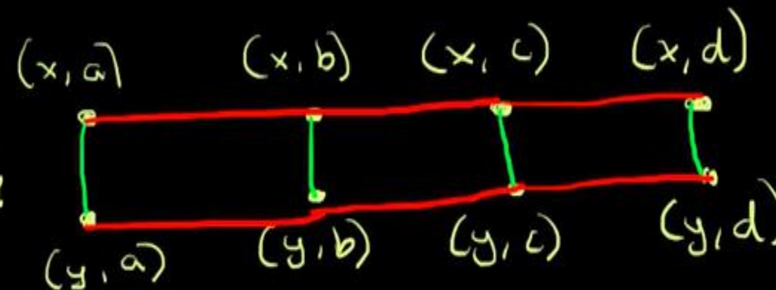
1) $i = k$ and $v_j v_l \in E(H)$

or 2) $j = l$ and $u_i u_k \in E(G)$

Ex: $K_2 \times P_4$

$K_2$ 


$P_4$ 


$K_2 \times P_4$ 


Ex: $K_4 \times K_3$


$$= \{ (u_i, v_j) \mid u_i \in V(G),\ v_j \in V(H) \}$$

and $e$ is an edge of $G \times H$ <u>iff</u> $e = (u_i, v_j)(u_k, v_\ell)$

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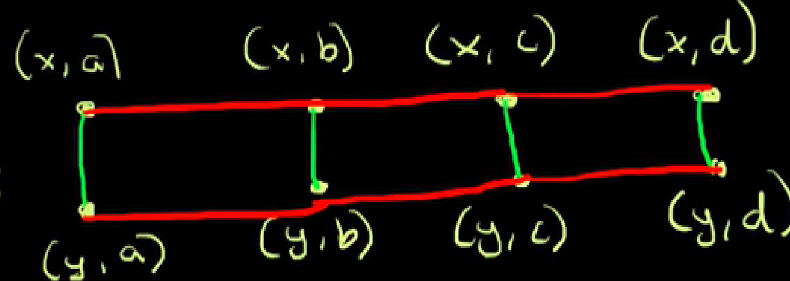
$K_2 \times P_4$ 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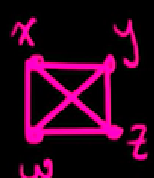
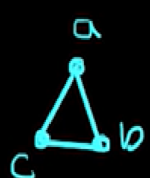
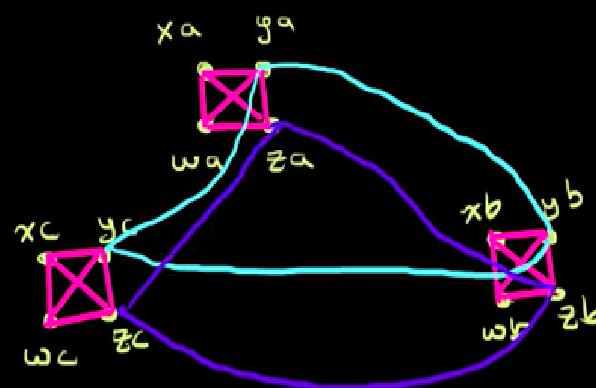


The hypercube (or n-cube) $Q_n$ is the graph

$$Q_n = \underbrace{K_2 \times K_2 \times \cdots \times K_2}_{n \text{ times}}$$

ie $Q_1 = K_2$

$Q_n = Q_{n-1} \times K_2$

$Q_1$

$Q_2$

$Q_3$

# Gray Codes

↳ A method for listing all 0-1 sequences of length $n$

Naive way: using the binary (base 2) representation of the integers $0, 1, 2, \dots, 2^n - 1$

Ex: $n = 3$

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 000 | 001 | 010 | 011 | 100 | 101 | 110 | 111 |

ie $3 = 2^1 + 2^0$ $\quad$ $6 = 2^2 + 2^1$

Another way: 000 001 101 100 110 111 011 010

This is an example of a Gray code

differ in exactly 1 position

Motivation:

011 → 100

all 3 electrical switches must change states

The second listing of these sequences requires only <u>one</u> switch to change states

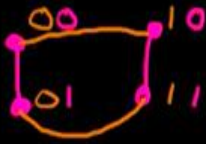
ie 101 → 100

A **Gray code** is a list of the $2^n$ binary sequences of length $n$ such that consecutive sequences in the list differ in exactly 1 position

A Gray code is called **cyclic** if the $1^{st}$ and last sequences in the list also differ in exactly 1 position.

Ex: $Q_1$


$Q_2$




The $n$-cube $Q_n$ is the graph whose vertices are labelled by the binary sequences of length $n$ where two vertices are adjacent if and only if their sequences differ in exactly 1 position

$Q_3$


$Q_n$ has a recursive structure


$A \cong Q_{n-1}$

$B \cong Q_{n-1}$

"corresponding" vertices are joined

A <u>Gray code</u> is a list of the $2^n$ binary sequences of length $n$ such that consecutive sequences in the list differ in exactly 1 position

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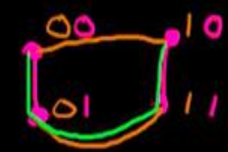
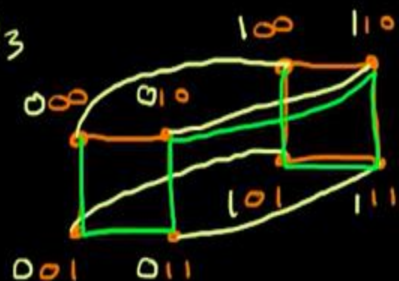



$Q_3$




$Q_n$ has a recursive structure


$A \cong Q_{n-1}$
$B \cong Q_{n-1}$

"corresponding" vertices are joined

A Gray code is equivalent to a Hamilton path in $Q_n$

# Binary Reflected Gray Codes

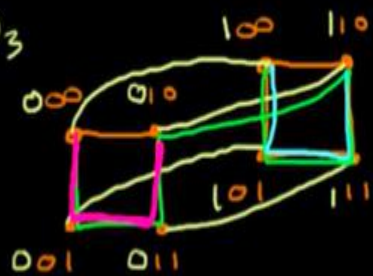
$Q_1$


0, 1

$Q_2$


00, 01, 11, 10

$Q_3$




000, 001, 011, 010, 110, 111, 101, 100

The Binary Reflected Gray Code (BRGC) is defined recursively as follows:

$$L_1 = 0, 1$$

$$L_n = 0 \cdot L_{n-1},\ 1 \cdot L_{n-1}^R \qquad \text{for } n \geq 2$$

where

$x \cdot L$ means concatenate $x$ at the beginning of every sequence of $L$

$L^R$ means $L$ in reverse order

Fact: For each integer $n \geq 1$ the BRGC is a Gray code

↳ Proof (induction on $n$)

Basis: $L_1$ is clearly a Gray code ✓

Ind. Hyp: Suppose for some $n \geq 2$ that $L_{n-1}$ is a Gray code

Now consider $L_n = 0 \cdot L_{n-1}, 1 \cdot L_{n-1}^R$

The sequences in $0 \cdot L_{n-1}$ still differ in 1 position (exactly) from their neighbours in the list since $L_{n-1}$ is a Gray code

Similarly for $1 \cdot L_{n-1}^R$

Also, the first sequence in $L_{n-1}^R$ is the last sequence in $L_{n-1}$

Thus the last sequence in $0 \cdot L_{n-1}$
& the first sequence in $1 \cdot L_{n-1}^R$
differ in exactly one position (the 1st position)

$\therefore L_n$ is a Gray code

Thank you

Let $u, v$ be vertices in a graph $G$.
The <u>distance</u> from $u$ to $v$ is the length of a shortest path from $u$ to $v$ in $G$ and is denoted $d(u,v)$

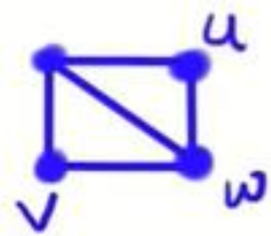
Sometimes denoted $d_G(u,v)$ for clarity

If $G$ is disconnected and $u, v$ are in different components we say $d(u,v) = \infty$

Ex:


$d(u,v) = 2$
$d(u,u) = 0$
$d(u,w) = 1$

Ex:




$d(u,v) = 2$
$d(u,u) = 0$
$d(u,w) = 1$

The distance is a metric space

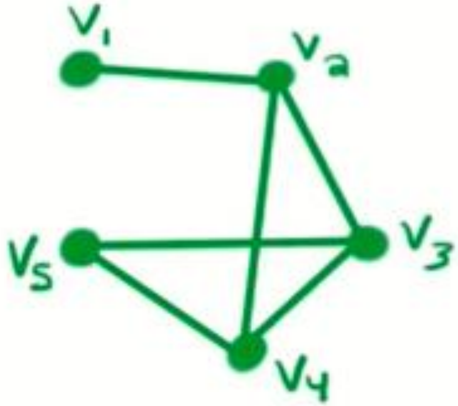
$$d: V \times V \to \mathbb{Z}^+ \cup \{0\}$$

1. $d(u,v) \geq 0$ and $d(u,v) = 0$ iff $u = v$
2. $d(u,v) = d(v,u)$
3. $d(u,v) + d(v,w) \geq d(u,w)$

The eccentricity of a vertex $v \in V(G)$ is

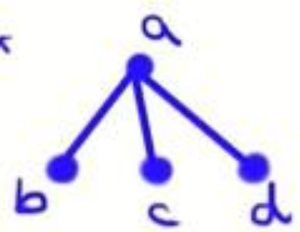
$$e(v) = \max\{d(u,v) \mid u \in V(G)\}$$

Ex




$e(v_1) = 3$
$e(v_2) = 2$
$e(v_3) = 2$
$e(v_4) = 3$
$e(v_5) = 3$

Ex



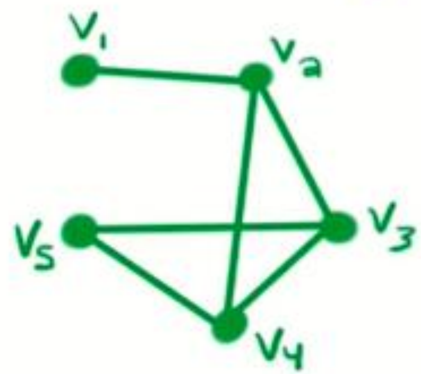

$e(a) = 1$
$e(b) = e(c) = e(d) = 2$

Note: $e(v) = 1 \iff v$ is adjacent to all other vertices

The eccentricity of a vertex $v \in V(G)$ is

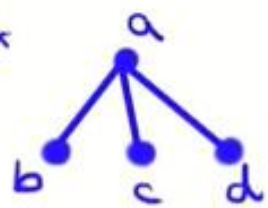
$$e(v) = \max \{ d(u,v) \mid u \in V(G) \}$$

Ex




$e(v_1) = 3$ diameter 3
$e(v_2) = 2$ radius 2
$e(v_3) = 2$
$e(v_4) = 3$
$e(v_5) = 3$

Ex

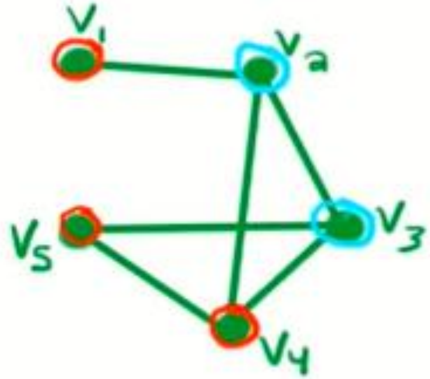



$e(a) = 1$ diameter 2
$e(b) = e(c) = e(d) = 2$ radius 1

Note: $e(v) = 1 \Leftrightarrow v$ is adjacent to all other vertices

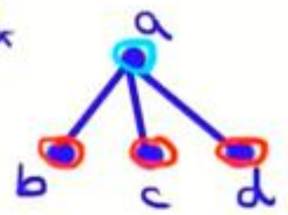
$$\mathrm{diam}(G) = \max \{ e(v) \mid v \in V(G) \}$$

$$\mathrm{rad}(G) = \min \{ e(v) \mid v \in V(G) \}$$

Ex 


$e(v_1) = 3$
$e(v_2) = 2$
$e(v_3) = 2$
$e(v_4) = 3$
$e(v_5) = 3$

diameter 3
radius 2

Ex 


$e(a) = 1$
$e(b) = e(c) = e(d) = 2$

diameter 2
radius 1

Note: $e(v) = 1 \iff$ v is adjacent to all other vertices

$\text{diam}(G) = \max\{e(v) \mid v \in V(G)\}$
$\text{rad}(G) = \min\{e(v) \mid v \in V(G)\}$

If $e(v) = \text{diam}(G)$ then v is a peripheral vertex.
The set of all such vertices make the periphery of G.

If $e(v) = \text{rad}(G)$ then v is a central vertex.
The set of all such vertices make the centre of G

Fact: $rad(G) \leq diam(G) \leq 2\,rad(G)$

obvious by definition

Proof: Let $u, v \in V(G)$ such that $d(u,v) = diam(G)$


Let $w$ be a central vertex of $G$

Then

$d(u,v) \leq d(u,w) + d(w,v)$

**Fact:** $\mathrm{rad}(G) \le \mathrm{diam}(G) \le 2\,\mathrm{rad}(G)$

obvious by definition

**Proof:** Let $u, v \in V(G)$ such that $d(u,v) = \mathrm{diam}(G)$


Let $w$ be a central vertex of $G$

Then

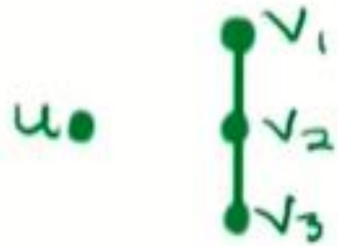
$$
\begin{aligned}
d(u,v) &\le d(u,w) + d(w,v) \\
&\le 2e(w) \\
&= 2\,\mathrm{rad}(G)
\end{aligned}
$$

$\square$

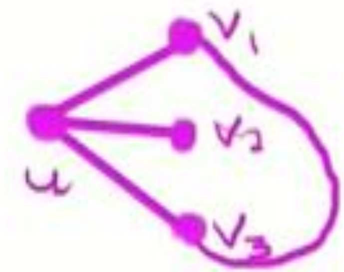
If $G$ is a disconnected graph, then $\overline{G}$ is connected and $\text{diam}(\overline{G}) \leq 2$

Ex:

$G$




$\overline{G}$



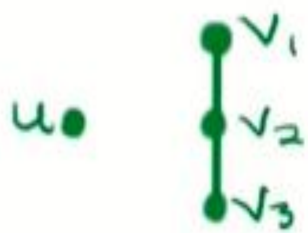

$e(u) = 1$

$e(v_1) = e(v_2)$

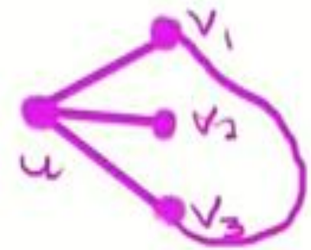
If $G$ is a disconnected graph, then $\overline{G}$ is connected and $\text{diam}(\overline{G}) \leq 2$

Ex:

$G$




$\overline{G}$




$e(u) = 1$

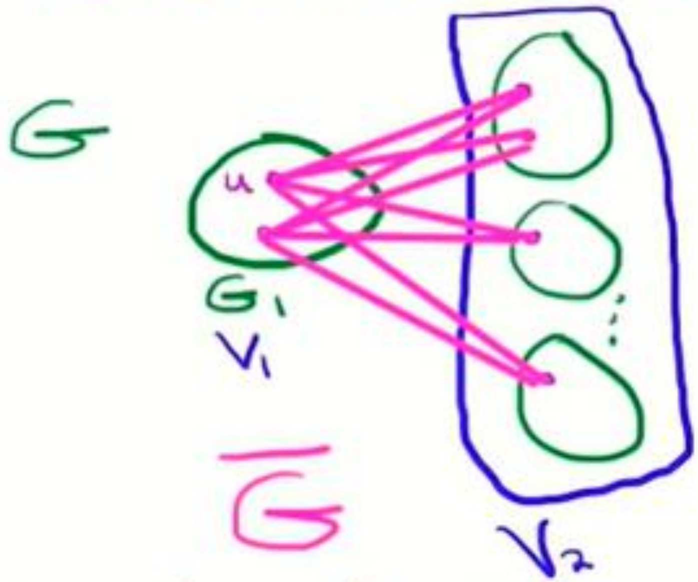
$e(v_1) = e(v_2) = e(v_3) = 2$

$\text{diam}(\overline{G}) = 2$

If $G$ is a disconnected graph, then $\overline{G}$ is connected and $\operatorname{diam}(\overline{G}) \le 2$

Proof:

Let $G$ be a disconnected graph and let $G_1$ be one of the connected components of $G$




Let $V_1 = V(G_1)$

$V_2 = V(G) \setminus V(G_1)$

Let $u \in V_1$

Then for every $v \in V_2$

$uv \notin E(G)$

$\therefore uv \in E(\overline{G})$ ie $d_{\overline{G}}(u,v) = 1$

And for every $v_1, v_2 \in V_2$

$d_{\overline{G}}(v_1, v_2) \le 2$

And for every $u_1, u_2 \in V_1$

$d_{\overline{G}}(u_1, u_2) \le 2$

$\therefore \max\{e_{\overline{G}}(v) \mid v \in V(\overline{G})\} \le 2$

ie $\operatorname{diam}(\overline{G}) \le 2$

Every graph $G$ is the centre of some connected graph.

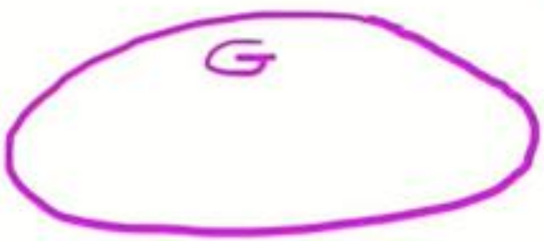
Recall: $Cen(G) = \{v \in V(G) \mid e(v) = rad(G)\}$

<u>Proof:</u>

Let $G$ be any graph.

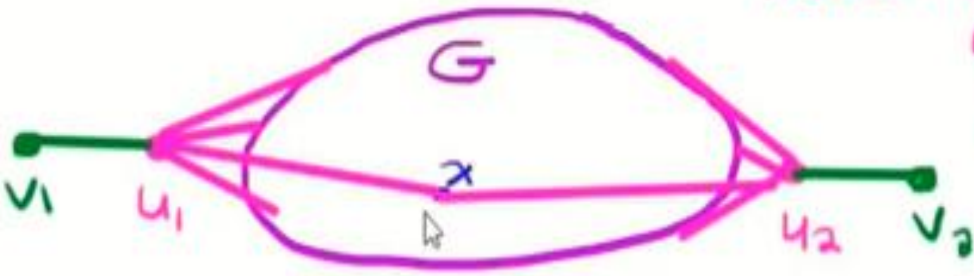
We construct a connected graph $H$ as follows:

Add 2 new vertices



Let G be any graph.

We construct a connected graph H as follows:




Add 2 new vertices to G

Call these $u_1$ & $u_2$

and add all edges

$u_1 x$ $\forall x \in V(G)$

and all edges

$u_2 x$ $\forall x \in V(G)$

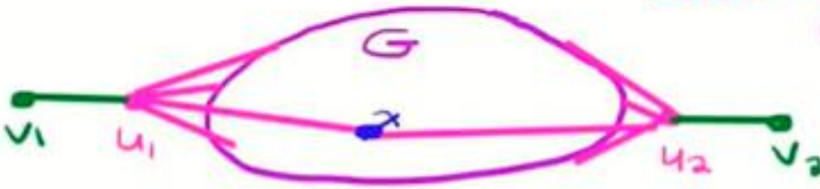
Next add 2 new vertices, $v_1$ and $v_2$ and edges $v_1 u_1$ and $v_2 u_2$

In the new graph H we have

$$e_H(v_1) = 4$$

Let $G$ be any graph.

We construct a connected graph $H$ as follows:




Add 2 new vertices to $G$

Call these $u_1$ & $u_2$

and add all edges

$u_1 x \quad \forall x \in V(G)$

and all edges

$u_2 x \quad \forall x \in V(G)$

Next add 2 new vertices, $v_1$ and $v_2$ and edges $v_1u_1$ and $v_2u_2$

In the new graph $H$ we have

$$e_H(v_1) = 4 \quad \& \quad e_H(v_2) = 4$$

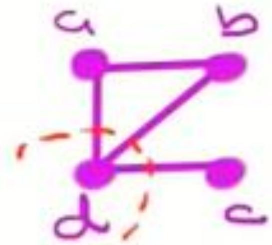
$$e_H(u_1) = 3 \quad \& \quad e_H(u_2) = 3$$

and $e_H(x) = 2 \quad \forall x \in V(G)$

$\therefore \ rad(H) = 2$ and $Cen(H) = \{v \in V(H) \mid e_H(v) = rad(H)\}$
$= V(G)$


connected


disconnected




connected
but easily made disconnected by removing vertex d

A vertex $v \in V(G)$ is called a <u>cut-vertex</u> if $G-v$ has more connected components than $G$

ie $c(G-v) > c(G)$

Notation: $c(G)$ = # of connected components of $G$

(sometimes $\kappa(G)$ is used)

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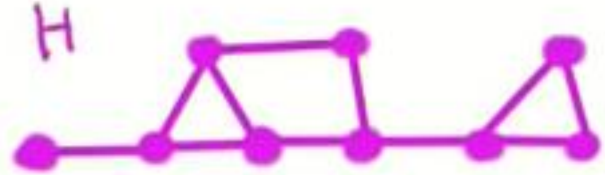
Notation: $c(G)$ = # of connected components of $G$

(Sometimes $\kappa(G)$ is used)

Ex: Find the cut-vertices:

G

H



A characterisation of cut-vertices:

Theorem: A vertex $v \in V(G)$ is a cut-vertex of $G$

$\iff \exists\ u, w \in V(G)$ $u, w \neq v$ such that $v$ is on every $u$-$w$ path of $G$.

Proof:

Assume $G$ is connected (otherwise just repeat the argument for each connected component)

$\Rightarrow$] Let $v \in V(G)$ be a cut-vertex

Then $G - v$ is disconnected.

Let $u, w$ be vertices in different components of $G - v$


So $\nexists$ any $u$-$w$ path in $G - v$

But $G$ is connected so $\exists$ $u$-$w$ paths in $G$ $\therefore$ all such paths went through vertex $v$

↳ Proof:

Assume $G$ is connected (otherwise just repeat the argument for each connected component)

$\Rightarrow$] Let $v \in V(G)$ be a cut-vertex

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So $\nexists$ any $u$-$w$ path in $G-v$

But $G$ is connected so $\exists$ $u$-$w$ paths in $G$ $\therefore$ all such paths went through vertex $v$


$\Leftarrow$] Suppose $\exists$ $u, w \in V(G)$ $u, w \neq v$ such that $v$ lies on every $u$-$w$ path

Then removing $v$ means $\nexists$ any $u$-$w$ path in $G-v$

$\therefore$ $G-v$ is disconnected

$\therefore$ $v$ is a cut-vertex

$\square$

How many cut-vertices can a connected graph have?


no cut-vertices


has $n-2$ cut-vertices

<u>Theorem</u>: Every non-trivial connected graph contains $\geq 2$ vertices that are not cut-vertices

**Theorem:** Every non-trivial connected graph contains $\geq 2$ vertices that are not cut-vertizes

Proof: (by contradiction)

Suppose the theorem is NOT true

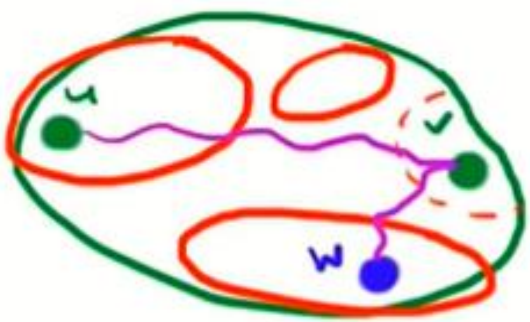
Then $\exists$ a non-trivial connected graph $G$ with at most 1 non-cut-vertex

(ie every vertex except possibly 1 is a cut-vertex)

Let $u, v \in V(G)$ with $d(u,v) = \text{diam}(G)$

At least one of $u, v$ is a cut-vertex, say $v$

So $G-v$ is disconnected




$G-v$

Let $w$ be a vertex in a different component of $G-v$ than $u$

Then every $u-w$ path contains $v$

$\therefore d(u,w) > d(u,v) = \text{diam}(G)$

$(\Rightarrow\Leftarrow)$

* This proof actually shows that no peripheral vertex is a cut-vertex

Thank you