Eccentricity, Radius, Diameter etc.

Let u, v be vertices in a graph G.
The <u>distance</u> from u to v is the length of a shortest path from u to v in G and is denoted $d(u,v)$

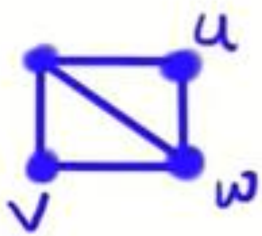
Sometimes denoted $d_G(u,v)$ for clarity

If G is disconnected and u, v are in different components we say $d(u,v) = \infty$

Ex:

$d(u,v) = 2$
$d(u,u) = 0$
$d(u,w) = 1$

Ex:



$d(u,v) = 2$
$d(u,u) = 0$
$d(u,w) = 1$

The distance is a metric space

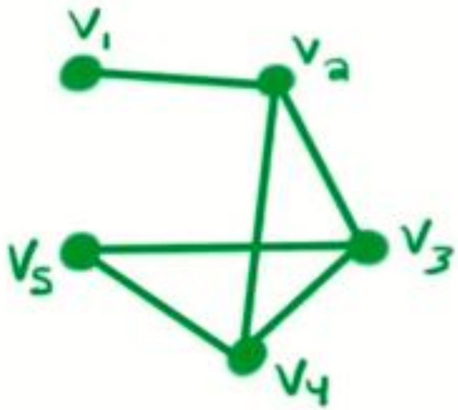
$$d: V \times V \to \mathbb{Z}^+ \cup \{0\}$$

1. $d(u,v) \geq 0$ and $d(u,v) = 0$ iff $u = v$
2. $d(u,v) = d(v,u)$
3. $d(u,v) + d(v,w) \geq d(u,w)$

The eccentricity of a vertex $v \in V(G)$ is

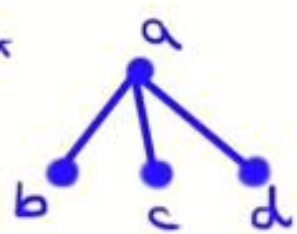
$$e(v) = \max\{d(u,v) \mid u \in V(G)\}$$

Ex



$e(v_1) = 3$
$e(v_2) = 2$
$e(v_3) = 2$
$e(v_4) = 3$
$e(v_5) = 3$

Ex



$e(a) = 1$
$e(b) = e(c) = e(d) = 2$

Note: $e(v) = 1 \iff v$ is adjacent to all other vertices

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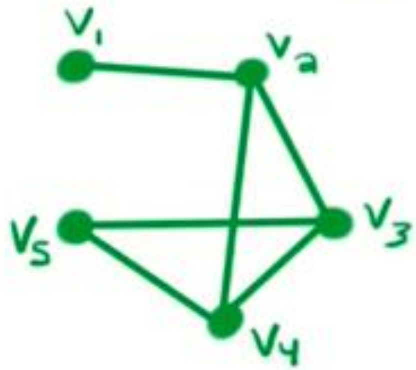
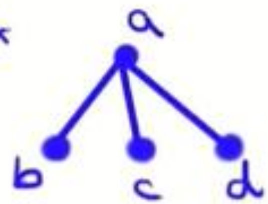
Ex



$e(v_1) = 3$
$e(v_2) = 2$
$e(v_3) = 2$
$e(v_4) = 3$
$e(v_5) = 3$

diameter 3
radius 2

Ex

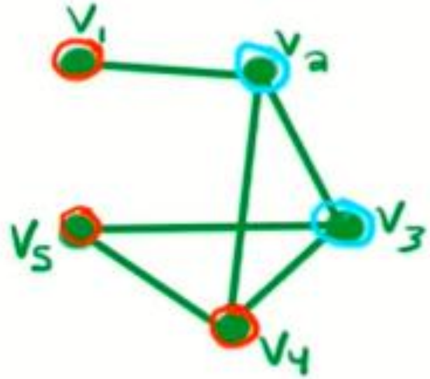


$e(a) = 1$
$e(b) = e(c) = e(d) = 2$

diameter 2
radius 1

Note: $e(v) = 1 \iff v$ is adjacent to all other vertices

$$\operatorname{diam}(G) = \max\{e(v) \mid v \in V(G)\}$$
$$\operatorname{rad}(G) = \min\{e(v) \mid v \in V(G)\}$$

Ex 

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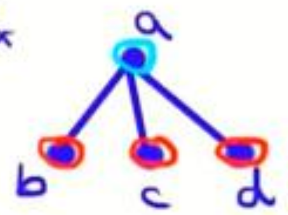
Ex 

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Note: $e(v) = 1 \iff v$ is adjacent to all other vertices

$\text{diam}(G) = \max\{e(v) \mid v \in V(G)\}$
$\text{rad}(G) = \min\{e(v) \mid v \in V(G)\}$

If $e(v) = \text{diam}(G)$ then v is a peripheral vertex.
The set of all such vertices make the periphery of G.

If $e(v) = \text{rad}(G)$ then v is a central vertex.
The set of all such vertices make the centre of G

Fact: $\mathrm{rad}(G) \le \mathrm{diam}(G) \le 2\,\mathrm{rad}(G)$

obvious by definition

Proof: Let $u, v \in V(G)$ such that $d(u,v) = \mathrm{diam}(G)$

Let w be a central vertex of G

Then

$$d(u,v) \le d(u,w) + d(w,v)$$

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Proof: Let $u, v \in V(G)$ such that
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Let w be a central vertex of G

Then

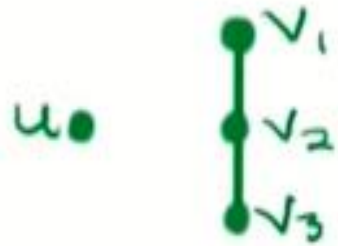
$$\begin{aligned} d(u,v) &\le d(u,w) + d(w,v) \\ &\le 2e(w) \\ &= 2\,\mathrm{rad}(G) \end{aligned}$$

$\square$

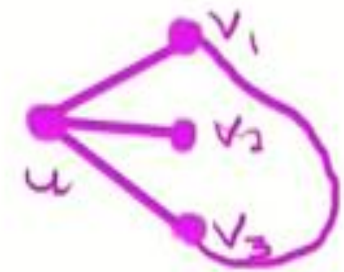
If G is a disconnected graph, then $\overline{G}$ is connected and $diam(\overline{G}) \leq 2$

Ex:

G



$\overline{G}$



$e(u)=1$

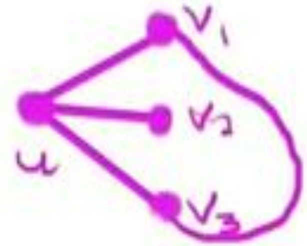
$e(v_1)=e(v_2)$

If G is a disconnected graph, then $\overline{G}$ is connected and $\operatorname{diam}(\overline{G}) \le 2$

Ex:

G

$\overline{G}$



$e(u) = 1$

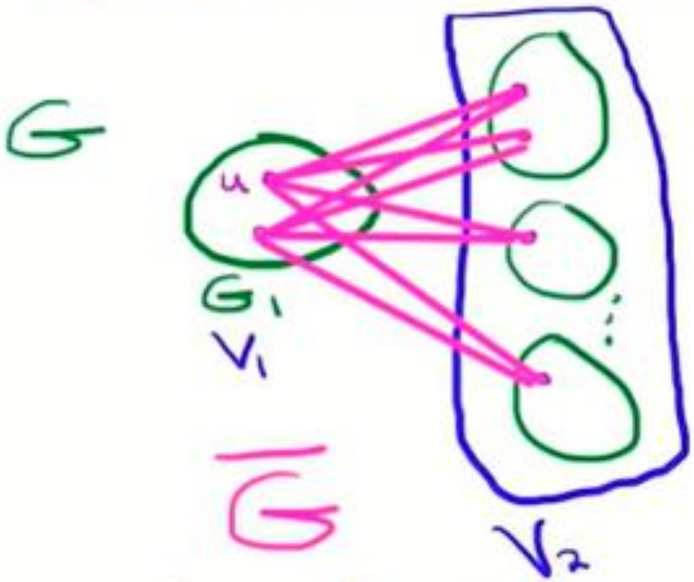
$e(v_1) = e(v_2) = e(v_3) = 2$

$\operatorname{diam}(\overline{G}) = 2$

If G is a disconnected graph, then $\overline{G}$ is connected and $\operatorname{diam}(\overline{G}) \leq 2$

Proof:

Let G be a disconnected graph and let G_1 be one of the connected components of G



Let $V_1 = V(G_1)$

$V_2 = V(G) \setminus V(G_1)$

Let $u \in V_1$

Then for every $v \in V_2$

$uv \notin E(G)$

$\therefore uv \in E(\overline{G})$ ie $d_{\overline{G}}(u,v) = 1$

And for every $v_1, v_2 \in V_2$

$d_{\overline{G}}(v_1, v_2) \leq 2$

And for every $u_1, u_2 \in V_1$

$d_{\overline{G}}(u_1, u_2) \leq 2$

$\overline{G}$

$\therefore \max\{e_{\overline{G}}(v) \mid v \in V(\overline{G})\} \leq 2$

ie $\operatorname{diam}(\overline{G}) \leq 2$

Every graph G is the centre of some connected graph.

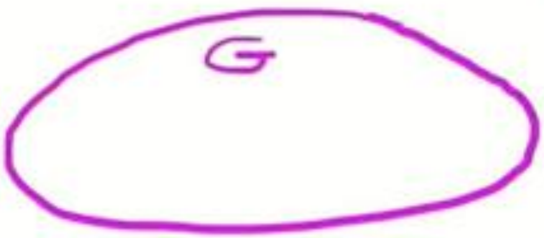
Recall: $Cen(G) = \{v \in V(G) \mid e(v) = rad(G)\}$

Proof:

Let G be any graph.

We construct a connected graph H as follows:

Add 2 new vertices



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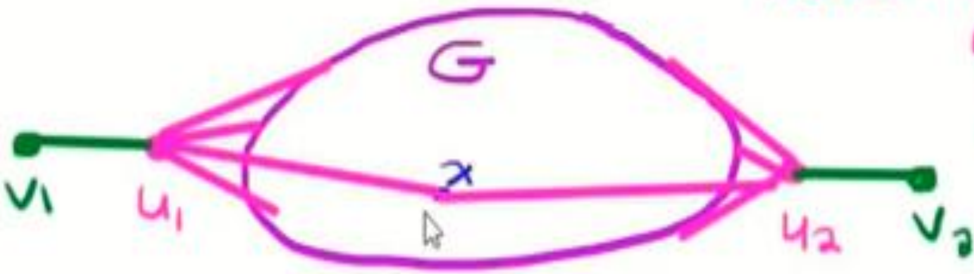


Add 2 new vertices to G

Call these u_1 & u_2

and add all edges

$u_1 x \quad \forall x \in V(G)$

and all edges

$u_2 x \quad \forall x \in V(G)$

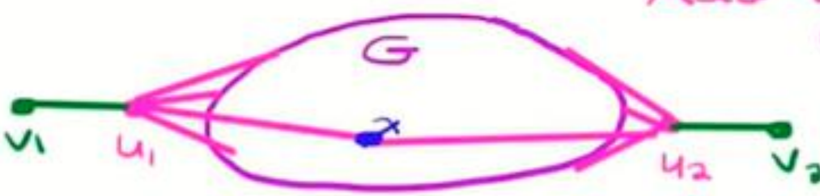
Next add 2 new vertices, v_1 and v_2 and edges $v_1 u_1$ and $v_2 u_2$

In the new graph H we have

$$e_H(v_1) = 4$$

Let G be any graph.

We construct a connected graph H as follows:



Add 2 new vertices to G

Call these u_1 & u_2

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$u_1 x \quad \forall x \in V(G)$

and all edges

$u_2 x \quad \forall x \in V(G)$

Next add 2 new vertices, v_1 and v_2 and edges $v_1 u_1$ and $v_2 u_2$

In the new graph H we have

$$e_H(v_1) = 4 \quad \& \quad e_H(v_2) = 4$$

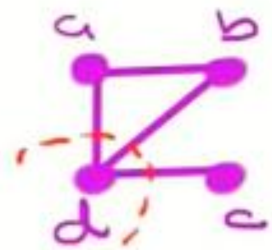
$$e_H(u_1) = 3 \quad \& \quad e_H(u_2) = 3$$

and $$e_H(x) = 2 \quad \forall x \in V(G)$$

$\therefore \operatorname{rad}(H) = 2$ and $\operatorname{Cen}(H) = \{v \in V(H) \mid e_H(v) = \operatorname{rad}(H)\}$
$= V(G)$

connected

disconnected



connected
but easily made disconnected by removing vertex d

A vertex $v \in V(G)$ is called a cut-vertex if $G-v$ has more connected components than G

ie $c(G-v) > c(G)$

Notation: $c(G)$ = # of connected components of G

(sometimes $\kappa(G)$ is used)

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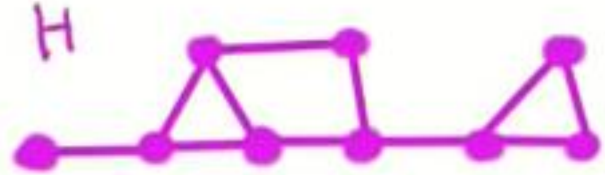
Ex: Find the cut-vertices:

G

H



A characterisation of cut-vertices:

<u>Theorem:</u> A vertex $v \in V(G)$ is a cut-vertex of G

$\iff \exists\, u, w \in V(G)$ $u, w \neq v$ such that v is on every u-w path of G.

Proof:

Assume G is connected (otherwise just repeat the argument for each connected component)

$\Rightarrow$] Let $v \in V(G)$ be a cut-vertex

Then $G - v$ is disconnected.

Let u, w be vertices in different components of $G - v$

So $\not\exists$ any u-w path in $G - v$

But G is connected so $\exists$ u-w paths in G $\therefore$ all such paths went through vertex v

↳ Proof:

Assume G is connected (otherwise just repeat the argument for each connected component)

$\Rightarrow$] Let $v \in V(G)$ be a cut-vertex

Then $G-v$ is disconnected.

Let u, w be vertices in different components of $G-v$

So $\nexists$ any u-w path in $G-v$

But G is connected so $\exists$ u-w paths in G $\therefore$ all such paths went through vertex v

$\Leftarrow$] Suppose $\exists\, u, w \in V(G)$ $u, w \neq v$ such that v lies on every u-w path

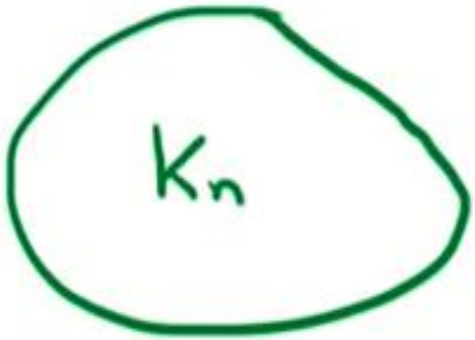
Then removing v means $\nexists$ any u-w path in $G-v$

$\therefore$ $G-v$ is disconnected

$\therefore$ v is a cut-vertex

$\square$

How many cut-vertices can a connected graph have?



no cut-vertices

has $n-2$ cut-vertices

Theorem: Every non-trivial connected graph contains ≥ 2 vertices that are not cut-vertices

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Proof: (by Contradiction)

Suppose the theorem is NOT true

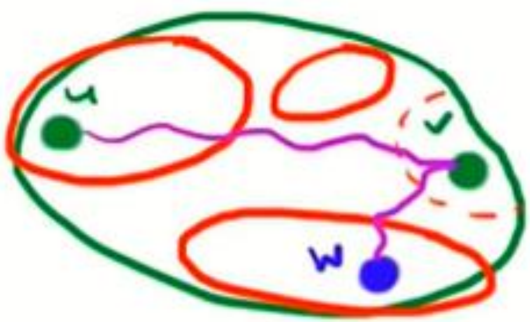
Then $\exists$ a non-trivial connected graph G with at most 1 non-cut-vertex

(ie every vertex except possibly 1 is a cut-vertex)

Let $u, v \in V(G)$ with $d(u,v) = \text{diam}(G)$

At least one of u, v is a cut-vertex, say v

So $G - v$ is disconnected



$G - v$

Let w be a vertex in a different component of $G - v$ than u

Then every u-w path contains v

$\therefore d(u,w) > d(u,v) = \text{diam}(G)$

($\Rightarrow\Leftarrow$)

* This proof actually shows that no peripheral vertex is a cut-vertex

Thank you