Decompositions

G

decompose into 3-cycles

edge-disjoint subgraph of G

A decomposition of a graph G is a family $\mathcal{F}$ of edge-disjoint subgraphs of G such that

Decompositions

$E(F_1) \cup E(F_2) = E(G)$

G

$\longrightarrow$ decompose into 3-cycles

F_1  F_2

edge-disjoint subgraph of G

$\mathcal{F} = \{F_1, F_2\}$

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path

If every graph of $\mathcal{F}$ is a cycle then the decomposition is called a **cycle decomposition**

path decomposition

There is a trivial path decomposition for every simple graph.

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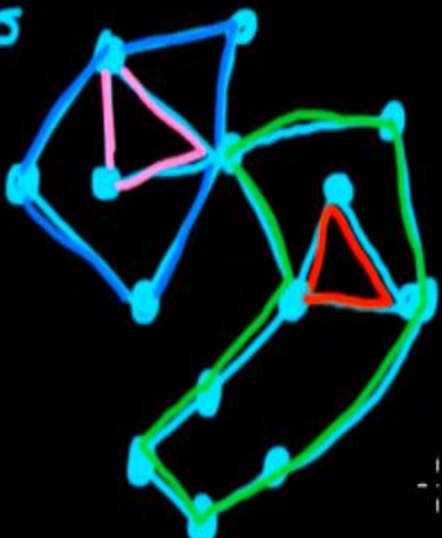
no cycle decomposition

G



$$\{C_3\ C_5\ C_9\ C_3\} = \mathcal{F}$$

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no cycle decomposition

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$$\{C_3 \; C_5 \; C_9 \; C_3\} = \mathcal{F}$$

$$\bigcup_{F \in \mathcal{F}} E(F) = E(G)$$

An <u>even graph</u> is a graph whose vertices all have even degree.

<u>Theorem</u> (Veblen 1912/13): A graph G admits a cycle decomposition $\Leftrightarrow$ G is even.

$\Rightarrow$] Suppose there is a cycle decomp of G

Take any $v \in V(G)$

either v is isolated, $d(v) = 0$

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$\Rightarrow$] Suppose there is a cycle decomp of G

Take any $v \in V(G)$

either v is isolated, $d(v) = 0$

or v belongs to some # of cycles in the decomposition k

Then $d(v) = 2k$

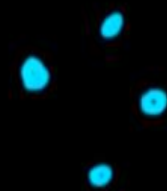
For all $v \in V(G)$ $d(v)$ is even

[$\Leftarrow$] Aim: G even $\Rightarrow$ G has a cycle decomp

*Induction on $|E(G)|$

Basis:

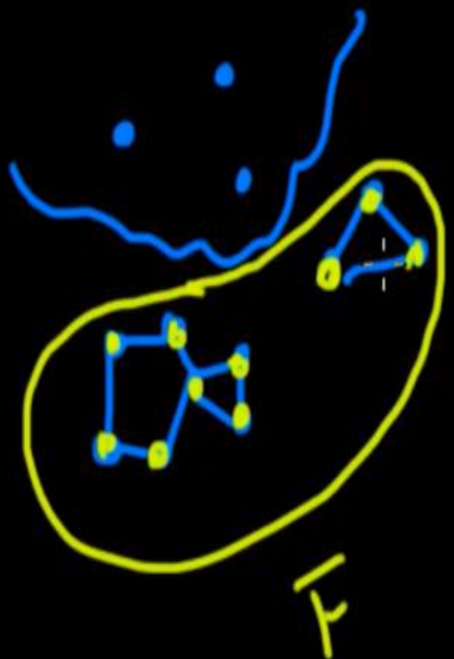
$|E(G)| = 0$ ✓



$\mathcal{F} = \{\ \}$

Ind. Hyp. Suppose that for all even graphs on $< m$ edges there exists a cycle decomp.

Take G to be even $|E(G)| = m$



X = the set of $V(G)$ with degree > 0

$F = G[X]$

an even graph

In F all vertices have degree ≥ 2

$\Rightarrow$ There is a cycle in F

F

an even graph

In F all vertices have degree ≥ 2

$\Rightarrow$ There is a cycle in F, call it C

Take $\underbrace{G'}_{\text{even graph}} = G \setminus E(C)$

G' is an even graph with $< m$ edges

By the Ind Hyp, G' has a cycle decomp, call it $\mathcal{C}'$

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$\underbrace{G'}_{\text{even graph}}$

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Now $\mathcal{C} = \mathcal{C}' \cup C$ is a cycle decomp of G.

$\square$

Eulerian $\Leftrightarrow$ even $\Leftrightarrow$ cycle decomposition

Euler tour

can have repeated vertices

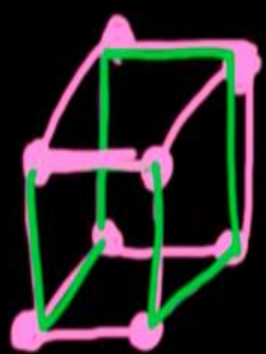
decompose $E(G)$ into $\{C_1, C_2, C_3, \ldots, C_k\}$ of any lengths

Hamiltonian Graphs

Hamilton path : path that covers every vertex once

Hamilton cycle : cycle that covers every vertex once

If $\exists$ Hamilton cycle the graph is <u>Hamiltonian</u>



Hamiltonian ✓

Ham. cycle can be converted to a Ham. path by removing one edge.

If G is Hamilton then any supergraph G'
$G \subseteq G'$ where G' is obtained by adding new edges between non-adjacent vertices of G then G' is also Hamilton.

Clear C_n is Hamiltonian $C_n = G$

$K_n = G'$

K_n is Hamiltonian.

1. Show that $K_{m,n}$ is Hamiltonian $\Leftrightarrow n = m \geq 2$

2. Give a graph that has no Hamilton path.

3. Can a Hamilton path always be used to form a Hamilton cycle? Explain.

① Show that $K_{m,n}$ is Hamiltonian $\Leftrightarrow n = m \geq 2$

$\Rightarrow$] Let $K_{m,n}$ have bipartition X, Y where $|X| = n$ $|Y| = m$

Let C be a Hamilton cycle of $K_{m,n}$

Each cycle in $K_{n,n}$ has even length and thus the cycle visits X and Y equally many times.

Then C visits X & Y equally many times and is incident with all vertices $\therefore |X| = |Y|$

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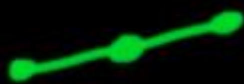
K_2

(2) Give a connected graph that has no Hamilton

G



no Ham. path

no Ham path

(3) Can a Hamilton path always be used

a Hamilton cycle? Explain.

② Give a connected graph that has no Hamilton path.

G

no Ham. path

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P_3

Lovász Conjecture: Every finite connected vertex transitive graph has a Hamilton path.

An automorphism of a graph G is an isomorphism from G to itself.

Vertices $u, v \in V(G)$ are similar if there is an automorphism α such that $\alpha(u) = v$.

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$$\alpha : V(G) \to V(G)$$
$$u \to v$$

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A vertex-transitive graph is a graph in which <u>all</u> vertices are similar.

v_1 and v_2

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$$v_2 \to v_1$$
$$v_3 \to v_3$$

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$v_1v_2 \in E(G)$ $\alpha(v_1)\alpha(v_2)$

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vertex-trans.

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vertex-trans.

vertex transitive graphs are regular

vertex-trans.

$v_3 \to v_3$

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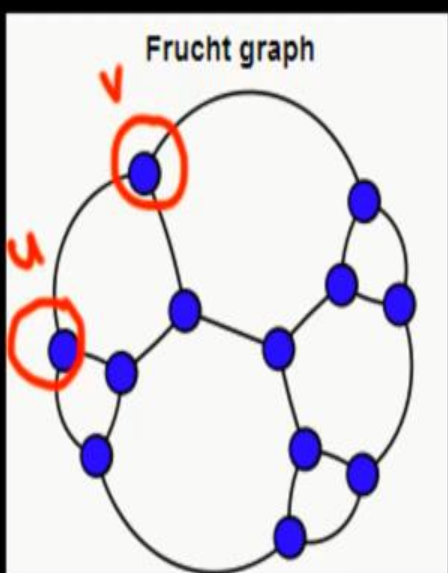
P_3

vertex-transitive graphs are <u>regular</u>



3-regular

<u>not</u> vertex-transitive

Thank you