Unit 3: Image Transforms (08)

2D-DFT, FFT, DCT, the KL Transform, Walsh / Hadamard Transform, Haar Transform, slant Transform , Basics of wavelet transform.

BASIC GEOMETRIC TRANSFORMATIONS

Transform theory plays a fundamental role in image processing, as working with the transform of an image instead of the image itself may give us more insight into the properties of the image. Two dimensional transforms are applied to image enhancement, restoration, encoding and description.

UNITARY TRANSFORMS

One dimensional signals

For a one dimensional sequence $\{f(x), 0 \le x \le N-1\}$ represented as a vector $\underline{f} = [f(0) f(1) \quad f(N-1)]^T$ of size N, a transformation may be written as

$$\underline{g} = \underline{T} \cdot \underline{f} \Rightarrow g(u) = \sum_{x=0}^{N-1} T(u, x) f(x), \; 0 \le u \le N-1$$

where $g(u)$ is the transform (or transformation) of $f(x)$, and $T(u, x)$ is the so called **forward transformation kernel**. Similarly, the inverse transform is the relation

$$f(x) = \sum_{u=0}^{N-1} I(x,u) g(u), \; 0 \le x \le N-1$$

or written in a matrix form

$$\underline{f} = \underline{I} \cdot \underline{g} = \underline{T}^{-1} \cdot \underline{g}$$

where $I(x,u)$ is the so called **inverse transformation kernel**.

If

$$I = \underline{T}^{-1} = \underline{T}^{*T}$$

the matrix $\underline{T}$ is called **unitary**, and the transformation is called unitary as well. It can be

proven (**how?**) that the columns (or rows) of an $N \times N$ unitary matrix are orthonormal and therefore, form a complete set of **basis vectors** in the N – dimensional vector space. In that case

$$\underline{f} = \underline{T}^{*T} \cdot \underline{g} \Rightarrow f(x) = \sum_{u=0}^{N-1} T^{*}(u, x) g(u)$$

The columns of $\underline{T}^{*T}$, that is, the vectors $\underline{T}^{*}{}_{u} = \left[T^{*}(u,0) \; T^{*}(u,1) \; T^{*}(u, N-1) \right]^{T}$ are called the **basis vectors** of $\underline{T}$.

Two dimensional signals (images)

As a one dimensional signal can be represented by an orthonormal set of **basis vectors**, an image can also be expanded in terms of a discrete set of **basis arrays** called basis images through a **two dimensional (image) transform**.

For an $N \times N$ image $f(x, y)$ the forward and inverse transforms are given below

$$g(u, v) = \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} T(u, v, x, y) f(x, y)$$

$$f(x, y) = \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} I(x, y, u, v) g(u, v)$$

where, again, $T(u, v, x, y)$ and $I(x, y, u, v)$ are called the **forward and inverse transformation kernels**, respectively.

If the kernel $T(u, v, x, y)$ of an image transform is separable and symmetric, then the transform $g(u, v) = \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} T(u, v, x, y) f(x, y) = \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} T_1(u, x) T_1(v, y) f(x, y)$ can be written in matrix form as follows

$$\underline{g} = \underline{T}_1 \cdot \underline{f} \cdot \underline{T}_1^T$$

The forward kernel is said to be **separable** if

$$T(u, v, x, y) = T_1(u, x) T_2(v, y)$$

It is said to be **symmetric** if T_1 is functionally equal to T_2 such that

$$T(u, v, x, y) = T_1(u, x) T_1(v, y)$$

The same comments are valid for the inverse kernel.

where f is the original image of size $N \times N$, and $\underline{T}_1$ is an $N \times N$ transformation matrix with elements $t_{ij} = T_1(i, j)$. If, in addition, $\underline{T}_1$ is a unitary matrix then the transform is called **separable unitary** and the original image is recovered through the relationship

$$\underline{f} = \underline{T}_1^{*T} \cdot \underline{g} \cdot \underline{T}_1^{*}$$

2D Discrete Fourier Transform

The independent variable (t,x,y) is discrete

$$F_r = \sum_{k=0}^{N_0-1} f[k] e^{-jr\Omega_0 k}$$

$$f_{N_0}[k] = \frac{1}{N_0} \sum_{r=0}^{N_0-1} F_r e^{jr\Omega_0 k}$$

$$\Omega_0 = \frac{2\pi}{N_0}$$

$$\Rightarrow$$

$$F[u,v] = \sum_{i=0}^{N_0-1} \sum_{k=0}^{N_0-1} f[i,k] e^{-j\Omega_0 (ui+vk)}$$

$$f_{N_0}[i,k] = \frac{1}{N_0^{\,2}} \sum_{u=0}^{N_0-1} \sum_{v=0}^{N_0-1} F[u,v] e^{j\Omega_0 (ui+vk)}$$

$$\Omega_0 = \frac{2\pi}{N_0}$$

Properties

- Linearity $\quad af(x,y) + bg(x,y) \Leftrightarrow aF(u,v) + bG(u,v)$
- Shifting $\quad f(x-x_0, y-x_0) \Leftrightarrow e^{-j2\pi(ux_0+vy_0)} F(u,v)$
- Modulation $\quad e^{j2\pi(u_0x+v_0y)} f(x,y) \Leftrightarrow F(u-u_0, v-v_0)$
- Convolution $\quad f(x,y) * g(x,y) \Leftrightarrow F(u,v)G(u,v)$
- Multiplication $\quad f(x,y)g(x,y) \Leftrightarrow F(u,v) * G(u,v)$
- Separability $\quad f(x,y) = f(x)f(y) \Leftrightarrow F(u,v) = F(u)F(v)$

Separability

1. Separability of the 2D Fourier transform
   - 2D Fourier Transforms can be implemented as a sequence of 1D Fourier Transform operations performed *independently* along the two axis

$$F(u,v)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi(ux+vy)}dxdy=$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi ux}e^{-j2\pi vy}dxdy=\int_{-\infty}^{\infty} e^{-j2\pi vy}dy\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi ux}dx=$$

$$=\int_{-\infty}^{\infty} F(u,y)e^{-j2\pi vy}dy=F(u,v)$$

2D DFT ⇨ 1D DFT along the rows → 1D DFT along the cols

- Separable functions can be written as $f(x,y)=f(x)g(y)$

2. The FT of a separable function is the product of the FTs of the two functions

$$F(u,v)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi(ux+vy)}dxdy=$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x)g(y)e^{-j2\pi ux}e^{-j2\pi vy}dxdy=\int_{-\infty}^{\infty} g(y)e^{-j2\pi vy}dy\int_{-\infty}^{\infty} h(x)e^{-j2\pi ux}dx=$$

$$=H(u)G(v)$$

$$f(x,y)=h(x)g(y)\Rightarrow F(u,v)=H(u)G(v)$$

WALSH TRANSFORM:

We define now the 1-D Walsh transform as follows:

$$W(u)=\frac{1}{N}\sum_{x=0}^{N-1} f(x)\left[\prod_{i=0}^{n-1}(-1)^{b_i(x)b_{n-1-i}(u)}\right]$$

The above is equivalent to:

$$W(u)=\frac{1}{N}\sum_{x=0}^{N-1}f(x)(-1)^{\sum_{i=1}^{n-1}b_i(x)b_{n-1-i}(u)}$$

The transform kernel values are obtained from:

$$T(u,x)=T(x,u)=\frac{1}{N}\left[\prod_{i=0}^{n-1}(-1)^{b_i(x)b_{n-1-i}(u)}\right]=\frac{1}{N}(-1)^{\sum_{i=1}^{n-1}b_i(x)b_{n-1-i}(u)}$$

Therefore, the array formed by the Walsh matrix is a real symmetric matrix. It is easily shown that it has orthogonal columns and rows

1-D Inverse Walsh Transform

$$f(x)=\sum_{x=0}^{N-1}W(u)\left[\prod_{i=0}^{n-1}(-1)^{b_i(x)b_{n-1-i}(u)}\right]$$

The above is again equivalent to

$$f(x)=\sum_{x=0}^{N-1}W(u)(-1)^{\sum_{i=1}^{n-1}b_i(x)b_{n-1-i}(u)}$$

The array formed by the inverse Walsh matrix is identical to the one formed by the forward Walsh matrix apart from a multiplicative factor N.

2-D Walsh Transform

We define now the 2-D Walsh transform as a straightforward extension of the 1-D transform:

$$W(u,v)=\frac{1}{N}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}f(x,y)\left[\prod_{i=0}^{n-1}(-1)^{b_i(x)b_{n-1-i}(u)+b_i(y)b_{n-1-i}(v)}\right]$$

•The above is equivalent to:

$$W(u,v)=\frac{1}{N}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}f(x,y)(-1)^{\sum_{i=1}^{n-1}(b_i(x)b_{n-1-i}(u)+b_i(x)b_{n-1-i}(u))}$$

Inverse Walsh Transform

We define now the Inverse 2-D Walsh transform. It is identical to the forward 2-D Walsh transform

$$f(x,y)=\frac{1}{N}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}W(u,v)\left[\prod_{i=0}^{n-1}(-1)^{b_i(x)b_{n-1-i}(u)+b_i(y)b_{n-1-i}(v)}\right]$$

•The above is equivalent to:

$$f(x,y)=\frac{1}{N}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}W(u,v)(-1)^{\sum_{i=1}^{n-1}(b_i(x)b_{n-1-i}(u)+b_i(x)b_{n-1-i}(u))}$$

HADAMARD TRANSFORM:

We define now the 2-D Hadamard transform. It is similar to the 2-D Walsh transform.

$$H(u,v)=\frac{1}{N}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}f(x,y)\left[\prod_{i=0}^{n-1}(-1)^{b_i(x)b_i(u)+b_i(y)b_i(v)}\right]$$

The above is equivalent to:

$$H(u,v)=\frac{1}{N}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}f(x,y)(-1)^{\sum_{i=1}^{n-1}(b_i(x)b_i(u)+b_i(x)b_i(u))}$$

We define now the Inverse 2-D Hadamard transform. It is identical to the forward 2-D Hadamard transform.

$$f(x,y)=\frac{1}{N}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}H(u,v)\left[\prod_{i=0}^{n-1}(-1)^{b_i(x)b_i(u)+b_i(y)b_i(v)}\right]$$

The above is equivalent to:

$$f(x,y)=\frac{1}{N}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}H(u,v)(-1)^{\sum_{i=1}^{n-1}(b_i(x)b_i(u)+b_i(x)b_i(u))}$$

DISCRETE COSINE TRANSFORM (DCT) :

The discrete cosine transform (DCT) helps separate the image into parts (or spectral sub-bands) of differing importance (with respect to the image's visual quality). The DCT is similar to the discrete Fourier transform: it transforms a signal or image from the spatial domain to the frequency domain.

The general equation for a 1D (*N* data items) DCT is defined by the following equation:

$$F(u) = \left(\frac{2}{N}\right)^{\frac{1}{2}} \sum_{i=0}^{N-1} \Lambda(i).cos\left[\frac{\pi.u}{2.N}(2i+1)\right] f(i)$$

and the corresponding *inverse* 1D DCT transform is simple $F^{-1}(u)$, i.e.:

where

$$\Lambda(i) = \begin{cases} \frac{1}{\sqrt{2}} & \text{for} \xi = 0 \\ 1 & \text{otherwise} \end{cases}$$

The general equation for a 2D (*N* by *M* image) DCT is defined by the following equation:

$$F(u,v) = \left(\frac{2}{N}\right)^{\frac{1}{2}} \left(\frac{2}{M}\right)^{\frac{1}{2}} \sum_{i=0}^{N-1} \sum_{j=0}^{M-1} \Lambda(i).\Lambda(j).cos\left[\frac{\pi.u}{2.N}(2i+1)\right] cos\left[\frac{\pi.v}{2.M}(2j+1)\right].f(i,j)$$

and the corresponding *inverse* 2D DCT transform is simple $F^{-1}(u,v)$, i.e.:

where

$$\Lambda(\xi) = \begin{cases} \frac{1}{\sqrt{2}} & \text{for} \xi = 0 \\ 1 & \text{otherwise} \end{cases}$$

The basic operation of the DCT is as follows:

- The input image is N by M;
- f(i,j) is the intensity of the pixel in row i and column j;
- F(u,v) is the DCT coefficient in row k1 and column k2 of the DCT matrix.
- For most images, much of the signal energy lies at low frequencies; these appear in the upper left corner of the DCT.
- Compression is achieved since the lower right values represent higher frequencies, and are often small - small enough to be neglected with little visible distortion.
- The DCT input is an 8 by 8 array of integers. This array contains each pixel's gray scale level;
- 8 bit pixels have levels from 0 to 255.

DISCRETE WAVELET TRANSFORM (DWT):

There are many discrete wavelet transforms they are Coiflet, Daubechies, Haar, Symmlet etc.

Haar Wavelet Transform

The Haar wavelet is the first known wavelet. The Haar wavelet is also the simplest possible wavelet. The Haar Wavelet can also be described as a step function f(x) shown in Eq

$$f(x) = \begin{cases} 1 & 0 \le x < 1/2, \\ -1 & 1/2 \le x < 1, \\ 0 & otherwise. \end{cases}$$

Each step in the one dimensional Haar wavelet transform calculates a set of wavelet coefficients (Hi-D) and a set of averages (Lo-D). If a data set $s_0, s_1, \ldots, s_{N-1}$ contains N elements, there will be N/2 averages and N/2 coefficient values. The averages are stored in the lower half of the N element array and the coefficients are stored in the upper half.

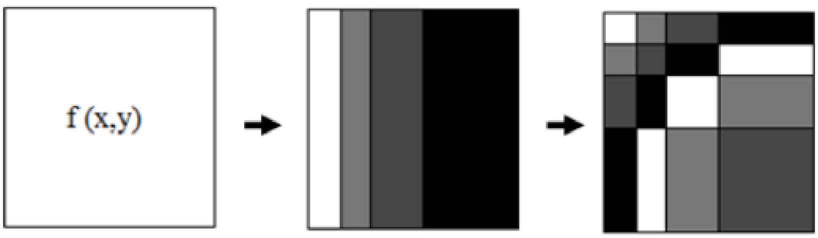
The Haar equations to calculate an average (a_i) and a wavelet coefficient (c_i) from the data set are shown below Eq

$$a_i = \frac{s_i + s_{i+1}}{2} \qquad c_i = \frac{s_i - s_{i+1}}{2}$$

In wavelet terminology the Haar average is calculated by the scaling function. The coefficient is calculated by the wavelet function.

Two-Dimensional Wavelets

The two-dimensional wavelet transform is separable, which means we can apply a one-dimensional wavelet transform to an image. We apply one-dimensional DWT to all rows and then one-dimensional DWTs to all columns of the result. This is called the standard decomposition and it is illustrated in figure 4.8.



Original Image DWT of rows DWT of columns

Figure The standard decomposition of the two-dimensional DWT.

We can also apply a wavelet transform differently. Suppose we apply a wavelet transform to an image by rows, then by columns, but using our transform at one scale only. This technique will produce a result in four quarters: the top left will be a half-sized version of the image and the other quarter's high-pass filtered images. These quarters will contain horizontal, vertical, and diagonal edges of the image. We then apply a one-scale DWT to the top-left quarter, creating smaller images, and so on. This is called the nonstandard decomposition, and is illustrated in figure 4.9.

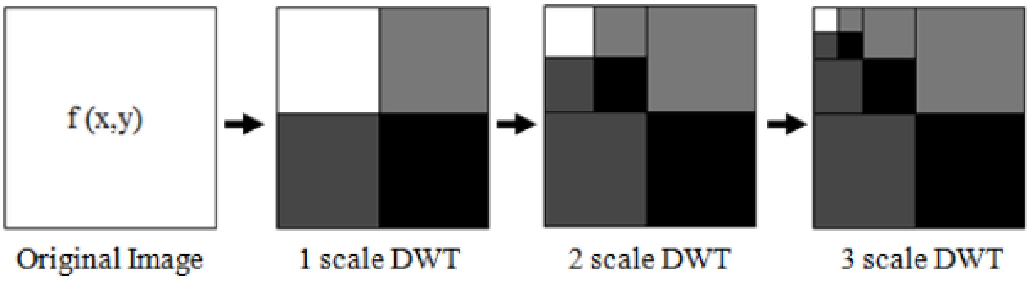


Figure 4.9 The nonstandard decomposition of the two-dimensional DWT.

Steps for performing a one-scale wavelet transform are given below:

Step 1: Convolve the image rows with the low-pass filter.

Step 2 : Convolve the columns of the result of step 1 with the low-pass filter and rescale this to half its size by sub-sampling.

Step 3 : Convolve the result of step 1 with high-pass filter and again sub-sample to obtain an image of half the size.

Step 4 : Convolve the original image rows with the high-pass filter.

Step 5: Convolve the columns of the result of step 4 with the low-pass filter and recycle this to half its size by sub-sampling.

Step 6 :Convolve the result of step 4 with the high-pass filter and again sub-sample to obtain an image of half the size.

At the end of these steps there are four images, each half the size of original. They are

1. The low-pass / low-pass image (LL), the result of step 2,
2. The low-pass / high-pass image (LH), the result of step 3,
3. The high-pass / low-pass image (HL), the result of step 5, and
4. The high-pass / high-pass image (HH), the result of step 6

These images can be placed into a single image grid as shown in the figure 4.10.

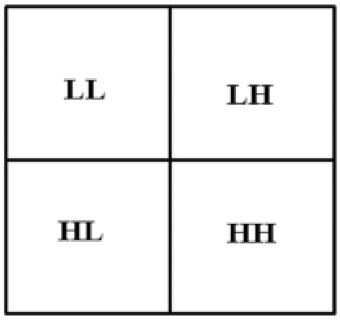


Figure 4.10 the one-scale wavelet transforms in terms of filters.

Figure 4.11 describes the basic dwt decomposition steps for an image in a block diagram form. The two-dimensional DWT leads to a decomposition of image into four components CA, CH, CV and CD, where CA are approximation and CH, CV, CD are details in three orientations (horizontal, vertical, and diagonal), these are same as LL, LH, HL, and HH. In these coefficients the watermark can be embedded.

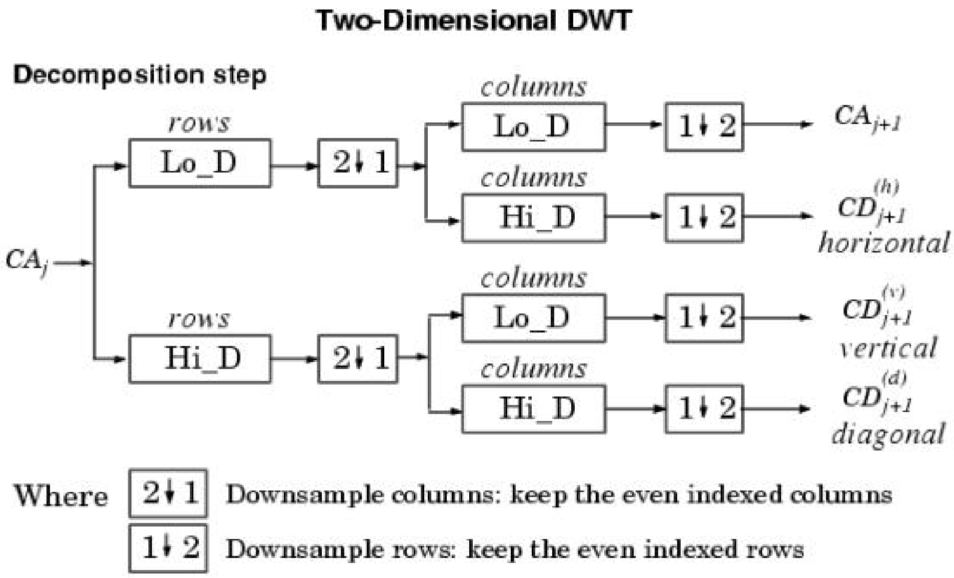
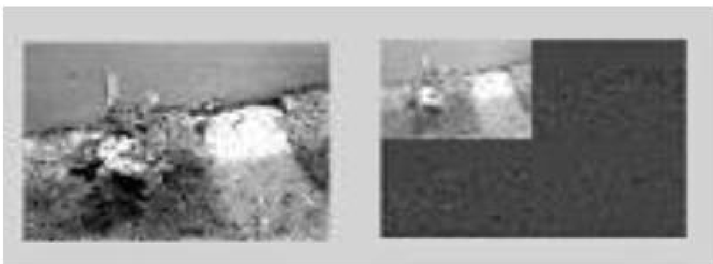


Figure 4.11 DWT decomposition steps for an image.



Original image and DWT decomposed image

An example of a discrete wavelet transform on an image is shown in Figure above. On the left is the original image data, and on the right are the coefficients after a single pass of the wavelet transform. The low-pass data is the recognizable portion of the image in the upper left corner. The high-pass components are almost invisible because image data contains mostly low frequency information.

The Kronecker Product

The Kronecker product has some of the same properties as conventional matrix multiplication. Both products follow the same properties for multiplication with a scalar. Also, both products are associative and they share the distributive property with conventional matrix addition. Furthermore, multiplying any matrix by the zero matrix yields the zero matrix. However, these two types of multiplication have many distinctions, such as results associated with taking transposes and inverses. Specifically, when taking the transpose or inverse of a conventional product of two matrices, the order of the matrices is reversed. In contrast, the transpose or inverse of a Kronecker product preserves the order of the two matrices.

Definition 1: Let $\mathbb{F}$ be a field. The *Kronecker product* of $A = \left[a_{ij}\right] \in M_{m,n}(\mathbb{F})$ and $B = \left[b_{ij}\right] \in M_{p,q}(\mathbb{F})$ is denoted by $A \otimes B$ and is defined to be the block matrix

$$A \otimes B \equiv \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix} \in M_{mp,nq}(\mathbb{F}).$$

Example 2: Let $A = \begin{pmatrix} 0 & -2 \\ 3 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 & 5 & 0 \\ -4 & -2 & 6 & 3 \\ -3 & 2 & -1 & 4 \end{pmatrix}$. Then,

$$A \otimes B = \begin{pmatrix} 0 & -2 \\ 3 & -1 \end{pmatrix} \otimes \begin{pmatrix} 2 & 1 & 5 & 0 \\ -4 & -2 & 6 & 3 \\ -3 & 2 & -1 & 4 \end{pmatrix}$$

$$
= \begin{pmatrix}
0 \cdot \begin{pmatrix} 2 & 1 & 5 & 0 \\ -4 & -2 & 6 & 3 \\ -3 & 2 & -1 & 4 \end{pmatrix} & -2 \cdot \begin{pmatrix} 2 & 1 & 5 & 0 \\ -4 & -2 & 6 & 3 \\ -3 & 2 & -1 & 4 \end{pmatrix} \\
3 \cdot \begin{pmatrix} 2 & 1 & 5 & 0 \\ -4 & -2 & 6 & 3 \\ -3 & 2 & -1 & 4 \end{pmatrix} & -1 \cdot \begin{pmatrix} 2 & 1 & 5 & 0 \\ -4 & -2 & 6 & 3 \\ -3 & 2 & -1 & 4 \end{pmatrix}
\end{pmatrix}
$$

$$
= \begin{pmatrix}
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} -4 & -2 & -10 & 0 \\ 8 & 4 & -12 & -6 \\ 6 & -4 & 2 & -8 \end{pmatrix} \\
\begin{pmatrix} 6 & 3 & 15 & 0 \\ -12 & -6 & 18 & 9 \\ -9 & 6 & -3 & 12 \end{pmatrix} & \begin{pmatrix} -2 & -1 & -5 & 0 \\ 4 & 2 & -6 & -3 \\ 3 & -2 & 1 & -4 \end{pmatrix}
\end{pmatrix}
$$

$$
= \begin{pmatrix}
0 & 0 & 0 & 0 & -4 & -2 & -10 & 0 \\
0 & 0 & 0 & 0 & 8 & 4 & -12 & -6 \\
0 & 0 & 0 & 0 & 6 & -4 & 2 & -8 \\
6 & 3 & 15 & 0 & -2 & -1 & -5 & 0 \\
-12 & -6 & 18 & 9 & 4 & 2 & -6 & -3 \\
-9 & 6 & -3 & 12 & 3 & -2 & 1 & -4
\end{pmatrix}
$$

In general, *A* ® *B* *- *B* ® *A* , even though both products result in matrices of the same size. This is seen in the next example where *B* ® *A* is calculated using the same matrices *A* and *B* from Example 2.

$$B \otimes A = \begin{pmatrix} 2 & 1 & 5 & 0 \\ -4 & -2 & 6 & 3 \\ -3 & 2 & -1 & 4 \end{pmatrix} \otimes \begin{pmatrix} 0 & -2 \\ 3 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -4 & 0 & -2 & 0 & -10 & 0 & 0 \\ 6 & -2 & 3 & -1 & 15 & -5 & 0 & 0 \\ 0 & 8 & 0 & 4 & 0 & -12 & 0 & -6 \\ -12 & 4 & -6 & 2 & 18 & -6 & 9 & -3 \\ 0 & 6 & 0 & -4 & 0 & 2 & 0 & -8 \\ -9 & 3 & 6 & -2 & -3 & 1 & 12 & -4 \end{pmatrix}$$

The Haar functions

The family of N Haar functions $h_k(t)$, $(k = 0, \cdots, N-1)$ are defined on the interval $0 \le t \le 1$. The shape of the specific function $h_k(t)$ of a given index k depends on two parameters p and q:

$$k = 2^p + q - 1$$

For any value of $k \geq 0$, p and q are uniquely determined so that 2^p is the largest power of 2 contained in k ($2^p < k$) and $q-1$ is the remainder $q - 1 = k - 2^p$. For example, when $N = 16$, the index k with the corresponding p and q are shown in the table:

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
p	0	0	1	1	2	2	2	2	3	3	3	3	3	3	3	3
q	0	1	1	2	1	2	3	4	1	2	3	4	5	6	7	8

Now the Haar functions can be defined recursively as:

- When $k = 0$, the Haar function is defined as a constant

$$h_0(t) = 1/\sqrt{N}$$

- When $k > 0$, the Haar function is defined by

$$h_k(t) = \frac{1}{\sqrt{N}} \begin{cases} 2^{p/2} & (q-1)/2^p \leq t < (q-0.5)/2^p \\ -2^{p/2} & (q-0.5)/2^p \leq t < q/2^p \\ 0 & \text{otherwise} \end{cases}$$

The Haar Transform Matrix

The N Haar functions can be sampled at $t = m/N$, where $m = 0, \cdots, N-1$ to form an N by N matrix for discrete Haar transform. For example, when $N = 2$, we have

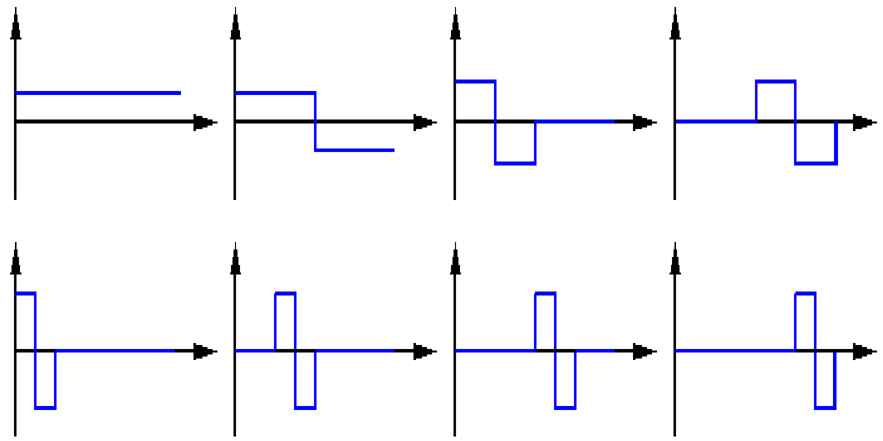
$$\mathbf{H}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

when $N = 4$, we have

$$\mathbf{H}_4 = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{bmatrix}$$

and when $N = 8$

$$\mathrm{H}_8 = \frac{1}{\sqrt{8}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} \\ 2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 \end{bmatrix} \begin{array}{l} \varphi_0(t) \\ \psi_0(t) \\ \psi_{1,0}(t) \\ \psi_{1,1}(t) \\ \psi_{2,0}(t) \\ \psi_{2,1}(t) \\ \psi_{2,2}(t) \\ \psi_{2,3}(t) \end{array}$$



We see that all Haar functions $h_k(t)$, $(k > 0)$ contains a single prototype shape composed of a square wave and its negative version, and the parameters

- p specifies the magnitude and width (or scale) of the shape;
- q specifies the position (or shift) of the shape.

Note that the functions $h_k(t)$ of Haar trnasform can represent not only the details in the signal of different scales (corresponding to different freq but also their locations in time.

The Haar transform matrix is real and orthogonal:

$$\mathbf{H} = \mathbf{H}^*, \quad \mathbf{H}^{-1} = \mathbf{H}^T, \quad \text{i.e.} \quad \mathbf{H}^T\mathbf{H} = \mathbf{I}$$

where $\mathbf{I}$ is identity matrix. For example, when $N = 4$,

$$\mathbf{H}_4^{-1}\mathbf{H}_4 = \mathbf{H}_4^T\mathbf{H}_4 = \frac{1}{4}\begin{bmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

In general, an N by N Haar matrix can be expressed in terms of its row vectors:

$$\mathbf{H} = \begin{bmatrix} \mathbf{h}_0^T \\ \mathbf{h}_1^T \\ \vdots \\ \mathbf{h}_{N-1}^T \end{bmatrix}, \quad \mathbf{H}^{-1} = \mathbf{H}^T = [\mathbf{h}_0, \cdots, \mathbf{h}_{\mathbf{N-1}}]$$

where $\mathbf{h}_n^T$ is the nth row vector of the matrix. The Haar tansform of a given signal vector $\mathbf{x} = [x[0], \cdots, x[N-1]]^T$ is

$$\mathbf{X} = \mathbf{H}\mathbf{x} = [\mathbf{h}_0, \cdots, \mathbf{h}_{\mathbf{N-1}}]\mathbf{x}$$

A Haar Transform Example:

The Haar transform coefficients of a $N = 4$-point signal $[x[0], x[1], x[2], x[3]]^T = [1, 2, 3, 4]^T$ can be found as

$$\frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ -1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

The inverse transform will express the signal as the linear combination of the basis functions:

$$\frac{1}{2}\begin{bmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{bmatrix}\begin{bmatrix} 5 \\ -2 \\ -1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$$= \frac{1}{2}[5\begin{bmatrix}1\\1\\1\\1\end{bmatrix} - 2\begin{bmatrix}1\\1\\-1\\-1\end{bmatrix} - \frac{1}{\sqrt{2}}\begin{bmatrix}\sqrt{2}\\-\sqrt{2}\\0\\0\end{bmatrix} - \frac{1}{\sqrt{2}}\begin{bmatrix}0\\0\\\sqrt{2}\\-\sqrt{2}\end{bmatrix}] = \begin{bmatrix}1\\2\\3\\4\end{bmatrix}$$

Note that coefficients $X[2] = -1/\sqrt{2}$ and $X[3] = -1/\sqrt{2}$ indicate not only there exist some detailed changes i

the signal, but also where in the signal such changes take place (first and second halves). This kind of position informa is not available in any other orthogonal transforms.

SLANT TRANSFORM

Slant transform is developed by Pratt et al and introduced by Enomoto and Shibata in 1971.

$[V] = [S_n][U][S_n]^r$

It is a 2D slant transform. Where, U is the original image of size N X N. and S_n is the unitary slant matrix.

This transform is a member of orthogonal transform. And for the first row it has a constant function and for the second row it has a linear function of the column index.

Properties of slant transform:

Real and orthogonal

Fast transform

Good energy compaction

Unitary kernel matrix starting with: $S_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

And iterating it according to the schema:

$$S_N = \frac{1}{\sqrt{2}}\left[\begin{array}{cc:c:cc:c} 1 & 0 & 0 & 1 & 0 & 0 \\ a_N & b_N & & -a_N & b_N & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \multicolumn{2}{c:}{0} & I_{(N/2)-2} & \multicolumn{2}{c:}{0} & I_{(N/2)-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & 0 & 0 & -1 & 0 \\ -b_N & a_N & & b_N & a_N & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \multicolumn{2}{c:}{0} & I_{(N/2)-2} & \multicolumn{2}{c:}{0} & -I_{(N/2)-2} \end{array}\right]\left[\begin{array}{c:c} S_{N/2} & 0 \\ \cdots & \cdots \\ 0 & S_{N/2} \end{array}\right]$$

Where I is the identity matrix of order $N/2-2$ and

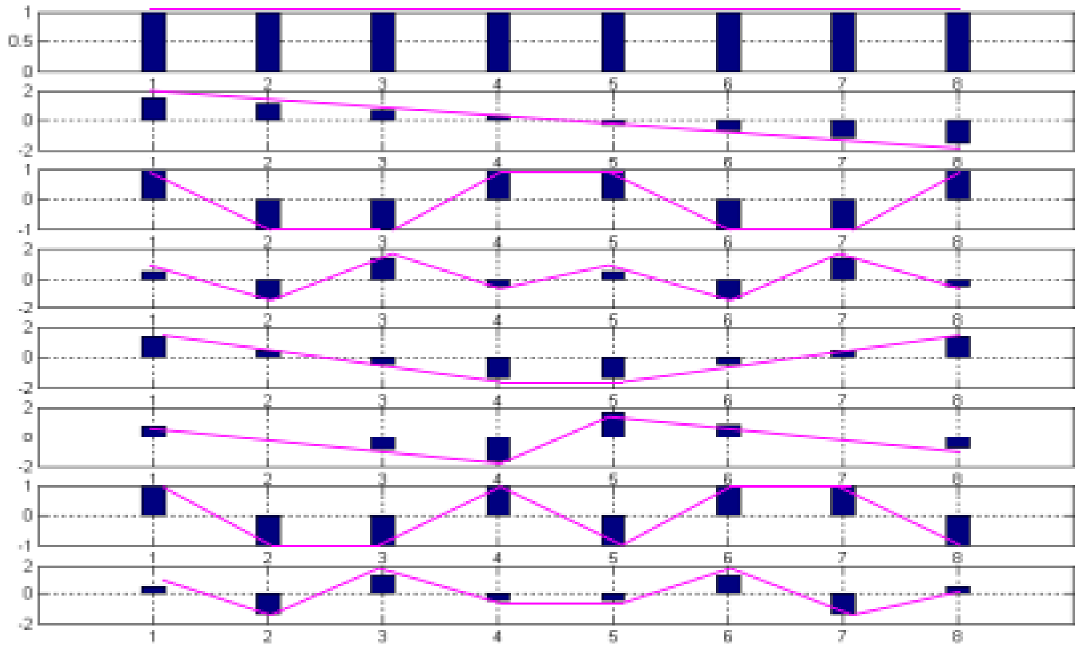
$$a_N = \left[\frac{3N^2}{4(N^2-1)}\right]^{1/2} \qquad b_N = \left[\frac{N^2-4}{4(N^2-1)}\right]^{1/2}$$

For example: $N=4$

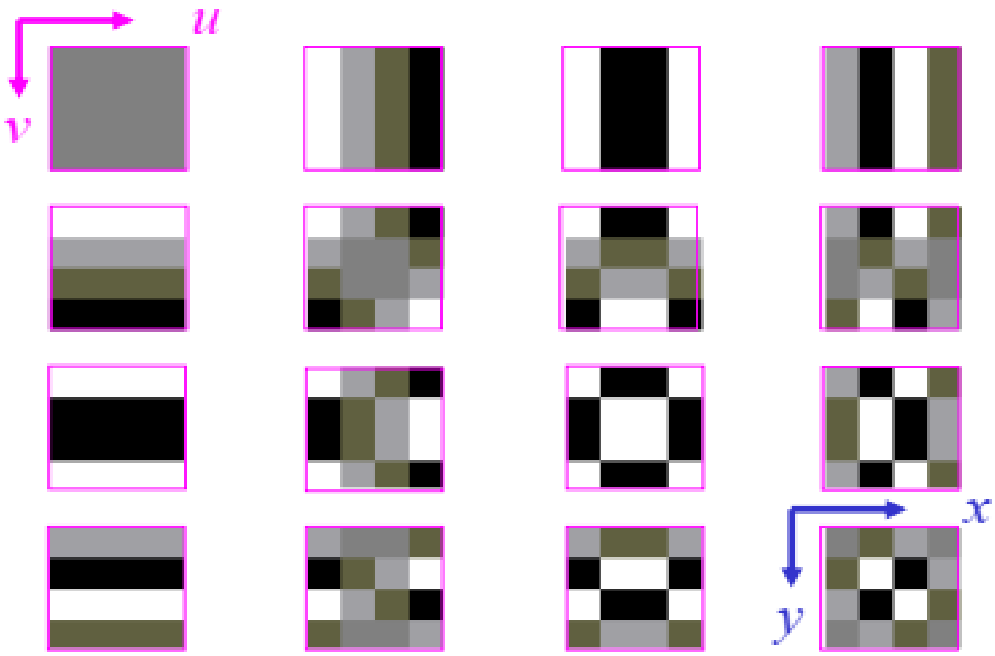
$$S_4 = \frac{1}{\sqrt{4}}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 3/\sqrt{5} & 1/\sqrt{5} & -1/\sqrt{5} & -3/\sqrt{5} \\ 1 & -1 & -1 & -1 \\ 1/\sqrt{5} & -3/\sqrt{5} & 3/\sqrt{5} & -1/\sqrt{5} \end{bmatrix}$$

For example: $N=8$

$$S_8 = \frac{1}{\sqrt{8}}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 7/\sqrt{21} & 5/\sqrt{21} & 3/\sqrt{21} & 1/\sqrt{21} & -1/\sqrt{21} & -3/\sqrt{21} & -5/\sqrt{21} & -7/\sqrt{21} \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1/\sqrt{5} & -3/\sqrt{5} & 3/\sqrt{5} & -1/\sqrt{5} & 1/\sqrt{5} & -3/\sqrt{5} & 3/\sqrt{5} & -1/\sqrt{5} \\ 3/\sqrt{5} & 1/\sqrt{5} & -1/\sqrt{5} & -3/\sqrt{5} & -3/\sqrt{5} & -1/\sqrt{5} & 1/\sqrt{5} & 3/\sqrt{5} \\ 7/\sqrt{105} & -1/\sqrt{105} & -9/\sqrt{105} & -17/\sqrt{105} & 17/\sqrt{105} & 9/\sqrt{105} & 1/\sqrt{105} & -7/\sqrt{105} \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1/\sqrt{5} & -3/\sqrt{5} & 3/\sqrt{5} & -1/\sqrt{5} & -1/\sqrt{5} & 3/\sqrt{5} & -3/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$$



The 1-D Slant transform basis functions for N=8



The 2-D Slant transform basis functions for N=4