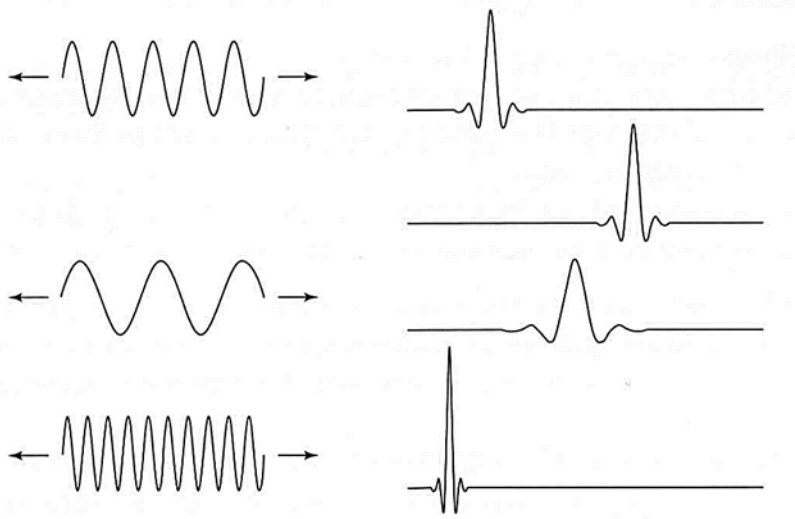
Wavelets and Multiresolution Processing

Wavelet Transform (WT)

Unlike Fourier transform whose basis functions are sinusoids, wavelet transform (WT) is based on wavelets. Wavelets (i.e. small waves) are mathematical functions that represent scaled and translated (shifted) copies of a finite-length waveform called the *mother wavelet* as shown in the figure below.



(a) (b)

Figure 9.1 Functions of (a) Fourier transform and (b) Wavelet transform

Wavelet transform is used to analyze a signal (image) into different frequency components at different resolution scales (i.e. multiresolution). This allows revealing image's spatial and frequency attributes simultaneously. In addition, features that might go undetected at one resolution may be easy to spot at another.

Multiresolution theory incorporates image pyramid and subband coding techniques.

Image Pyramid

is a powerful simple structure for representing images at more than one resolution. It is a collection of decreasing resolution images arranged in the shape of a pyramid as shown in the figure below.

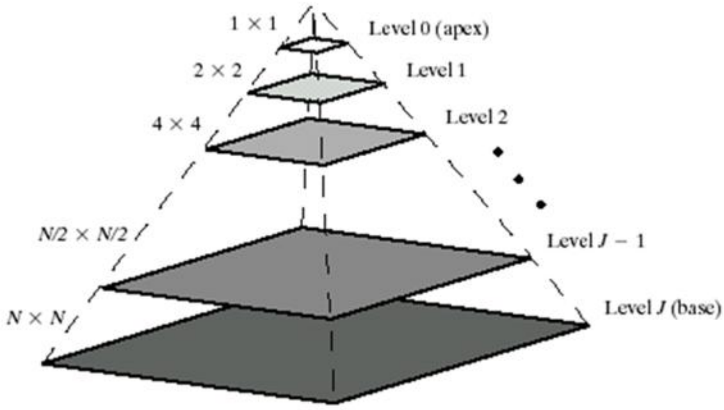


Figure 9.2 Image pyramid

The base of the pyramid contains a high-resolution representation of the image being processed; the apex contains a low-resolution approximation. As we move up the pyramid, both size and resolution decrease.

Subband Coding

is used to decompose an image into a set of bandlimited components called subbands, which can be reassembled to reconstruct the original image without error. Each subband is generated by bandpass filtering the input image. The next figures show 1D and 2D subband coding.

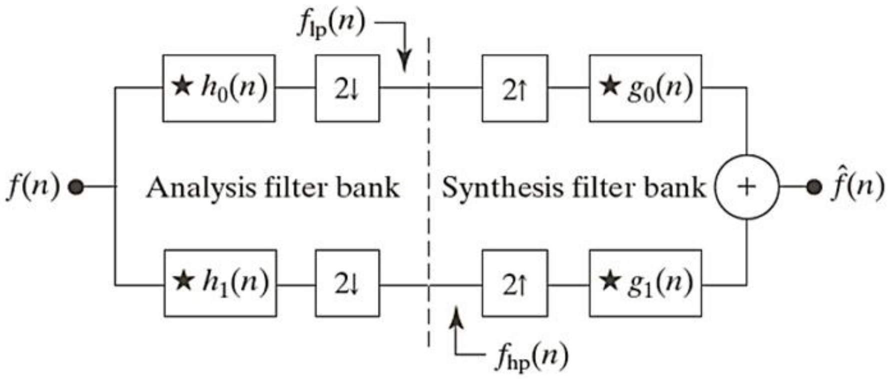


Figure 9.3 1D Two-band subband coding and decoding system

The analysis filter bank consists of:

- Lowpass filter $h_0(n)$ whose output, subband $f_{lp}(n)$ is called *approximation* subband of $f(n)$
- Highpass filter $h_1(n)$ whose output subband $f_{hp}(n)$ is called high frequency or *detail* part of $f(n)$

The synthesis bank filters $g_0(n)$ and $g_1(n)$ combine $f_{lp}(n)$ and $f_{hp}(n)$ to produce $f^{\wedge}(n)$.

The 2D subband coding is shown in the figure below.

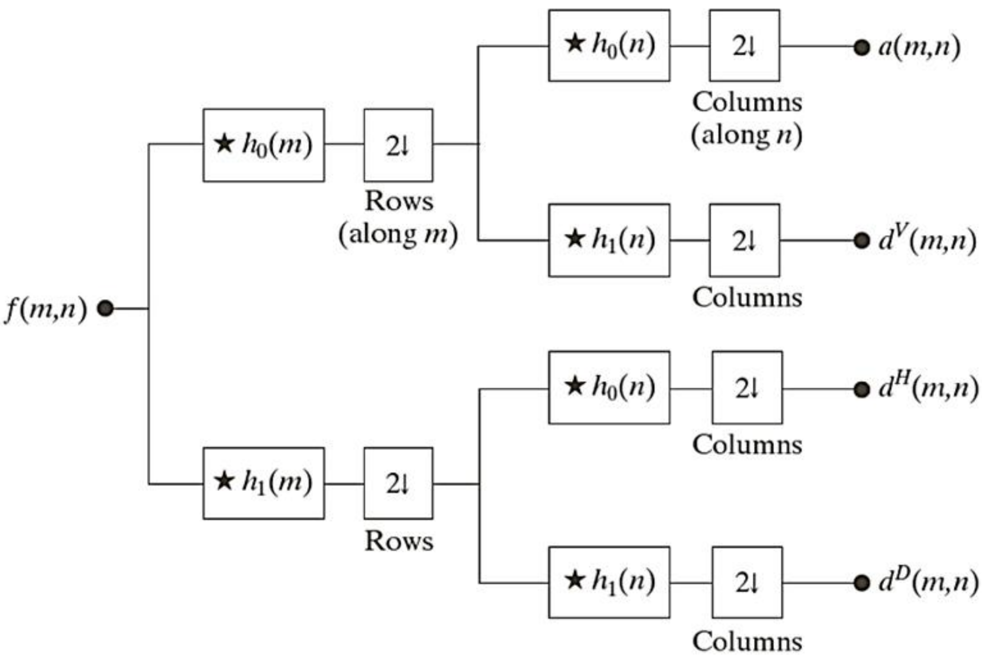


Figure 9.4 2D Four-band subband image coding

2D-Discrete Wavelet Transform (2D-DWT)

The DWT provides a compact representation of a signal's frequency components with strong spatial support. DWT decomposes a signal into frequency subbands at different scales from which it can be perfectly reconstructed.

2D-signals such as images can be decomposed using many wavelet decomposition filters in many different ways. We study the Haar wavelet filter and the pyramid decomposition method.

The Haar Wavelet Transform (HWT)

The Haar wavelet is a discontinuous, and resembles a step function.
For a function **f**, the HWT is defined as:

$$\mathrm{f} \rightarrow (\mathrm{a}^{\mathrm{L}} \mid \mathrm{d}^{\mathrm{L}})$$

$$\mathrm{a}^{\mathrm{L}} = (\mathrm{a}_1, \mathrm{a}_2, \dots, \mathrm{a}_{\mathrm{N}/2})$$

$$\mathrm{d}^{\mathrm{L}} = (\mathrm{d}_1, \mathrm{d}_2, \dots, \mathrm{d}_{\mathrm{N}/2})$$

where **L** is the decomposition level, **a** is the *approximation* subband and **d** is the *detail* subband.

$$\mathrm{a}_{\mathrm{m}} = \frac{\mathrm{f}_{2\mathrm{m}} + \mathrm{f}_{2\mathrm{m}-1}}{\sqrt{2}} \qquad for\ m = 1,2,\dots,N/2$$

$$\mathrm{d}_{\mathrm{m}} = \frac{\mathrm{f}_{2\mathrm{m}} - \mathrm{f}_{2\mathrm{m}-1}}{\sqrt{2}} \qquad for\ m = 1,2,\dots,N/2$$

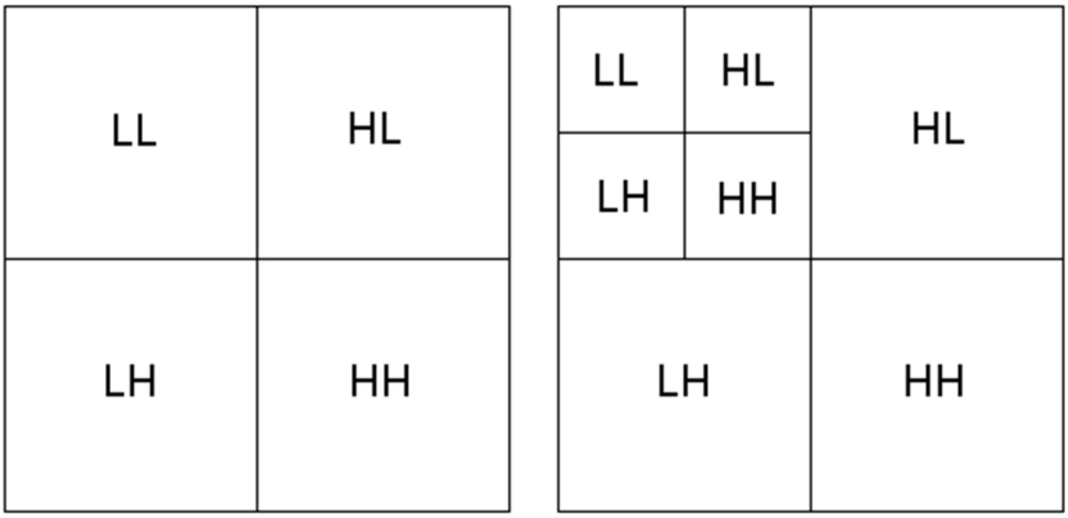
For example, if **f**$=\{f_1,f_2,f_3,f_4,f_5,f_6,f_7,f_8\}$ is a time-signal of length 8, then the HWT decomposes **f** into an approximation subband containing the Low frequencies and a detail subband containing the high frequencies:

$$\text{Low} = \mathrm{a} = \{f_2 + f_1,\ f_4 + f_3,\ f_6 + f_5,\ f_8 + f_7\}/\sqrt{2}$$

$$\text{High} = \mathrm{d} = \{f_2 - f_1,\ f_4 - f_3,\ f_6 - f_5,\ f_8 - f_7\}/\sqrt{2}$$

To apply HWT on images, we first apply a one level Haar wavelet to each row and secondly to each column of the resulting "image" of the first operation. The resulted image is decomposed into four subbands: LL, HL, LH, and HH subbands. (L=Low, H=High). The LL-subband contains an approximation of the original image while the other subbands contain the missing details. The LL-subband output from any stage can be decomposed further.

The figure below shows the result of one and two level HWT based on the pyramid decomposition

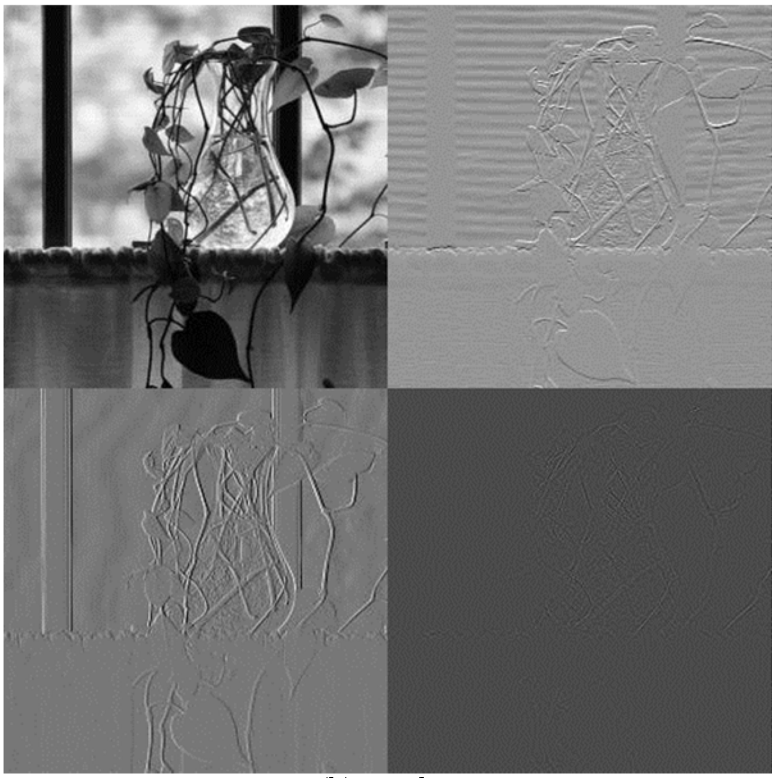


(a) Decomposition Level 1 (b) Decomposition Level 2

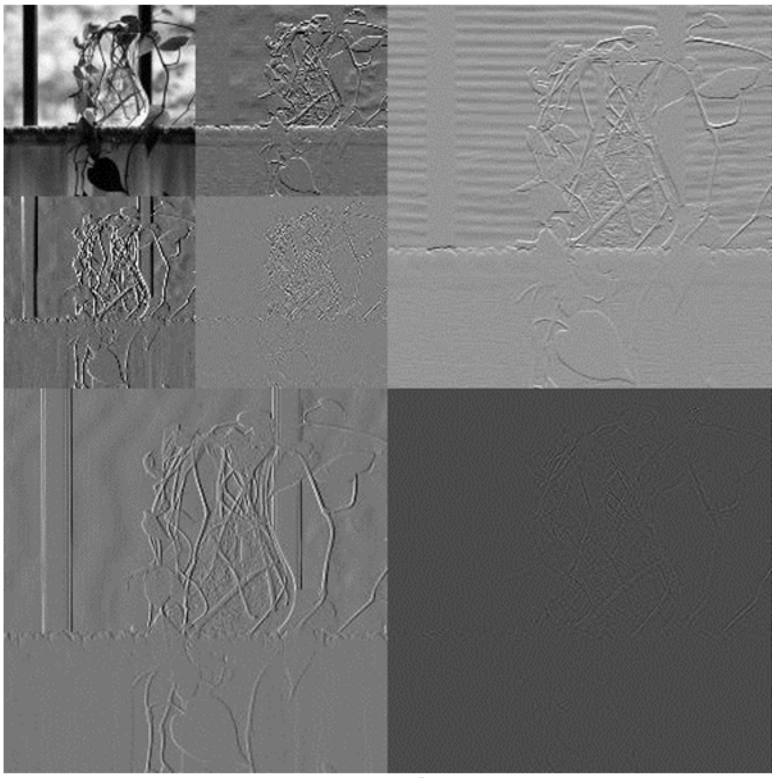
Figure 9.5 Pyramid decomposition using Haar wavelet filter

The next figure shows an image decomposed with 3-level Haar wavelet transform.

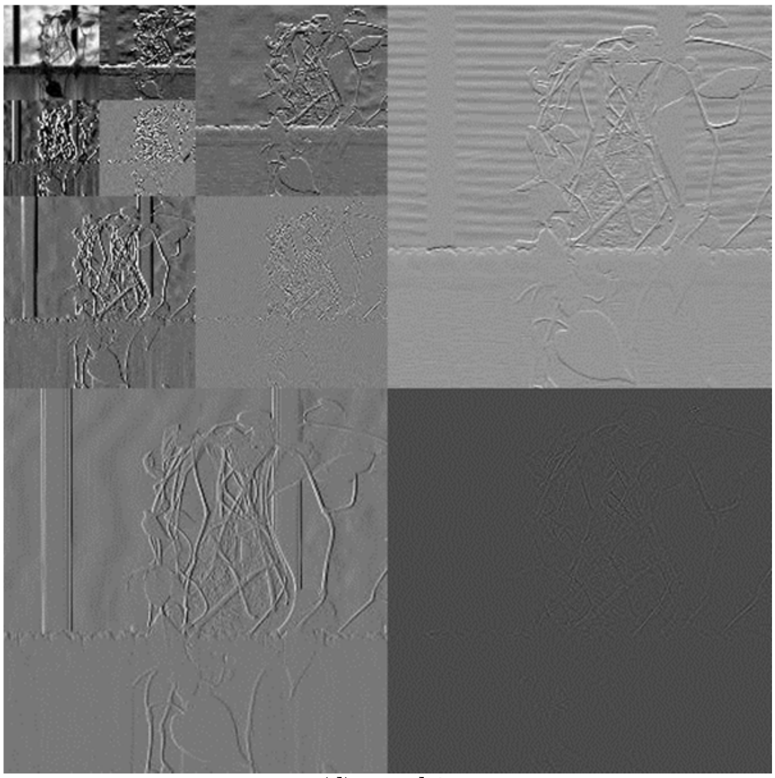
(a) Original image



(b) Level 1



(c) Level 2



(d) Level 3

Figure 9.6 Example of a Haar wavelet transformed image

Wavelet transformed images can be perfectly reconstructed using the four subbands using the inverse wavelet transform.

Inverse Haar Wavelet Transform (IHWT)

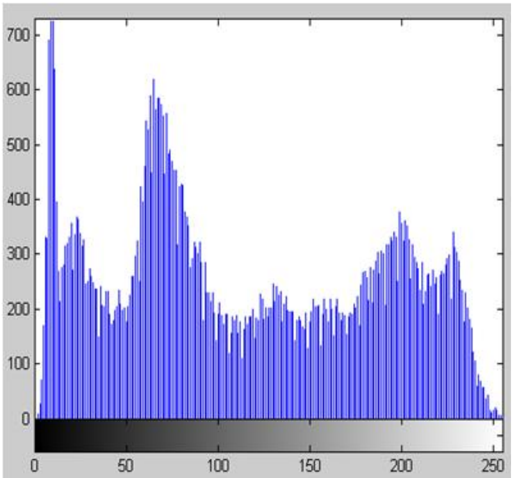
The inverse of the Haar wavelet transform is computed in the reverse order as follows:

$$\mathbf{f} = \left(\frac{a_1 - d_1}{\sqrt{2}}, \frac{a_1 + d_1}{\sqrt{2}}, \ldots, \frac{a_{N/2} - d_{N/2}}{\sqrt{2}}, \frac{a_{N/2} + d_{N/2}}{\sqrt{2}}\right)$$

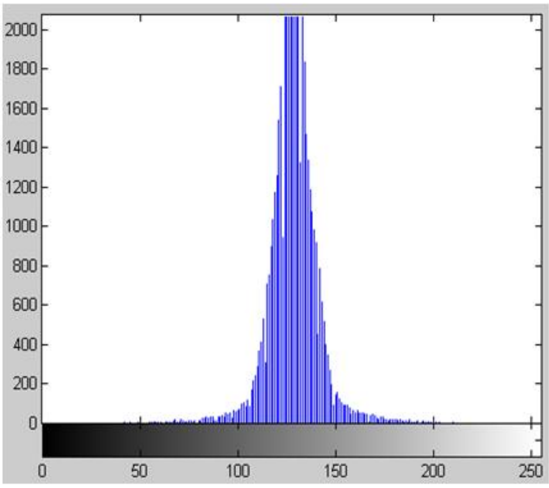
To apply IHWT on images, we first apply a one level inverse Haar wavelet to each column and secondly to each row of the resulting "image" of the first operation.

Statistical Properties of Wavelet subbands

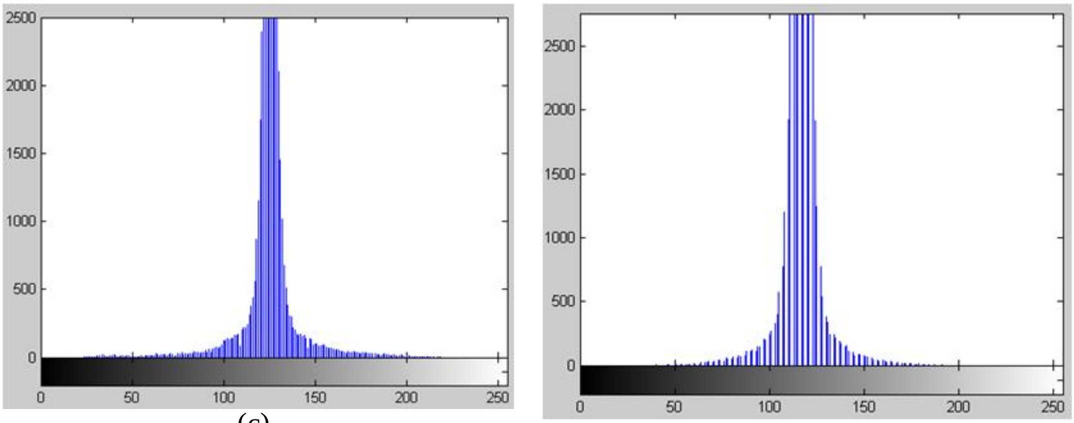
The distribution of the LL-subband approximates that of the original image but all non-LL subbands have a Laplacian distribution. This remains valid at all depths (i.e. decomposition levels).



(a)



(b)



(c) (d)

Figure 9.7 Histogram of (a) LL-subband (b) HL-subband (c) LH-subband (d) HH-subband of subbands in Figure 9.6 (b)

Wavelet Transforms in image processing

Any wavelet-based image processing approach has the following steps:

1. Compute the 2D-DWT of an image
2. Alter the transform coefficients (i.e. subbands)
3. Compute the inverse transform

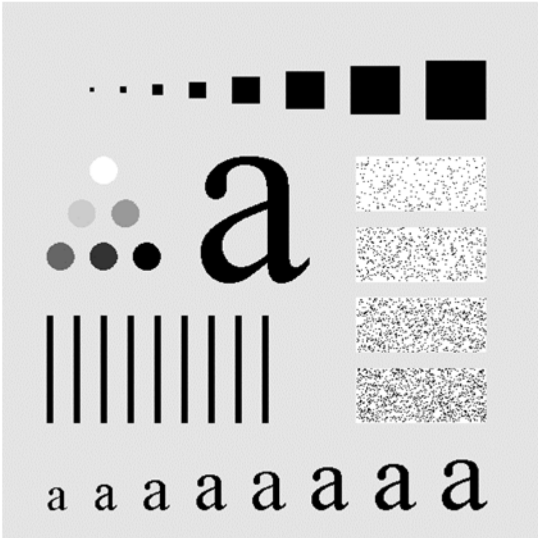
Wavelet transforms are used in a wide range of image applications. These include:

- Image and video compression
- Feature detection and recognition
- Image denoising
- Face recognition

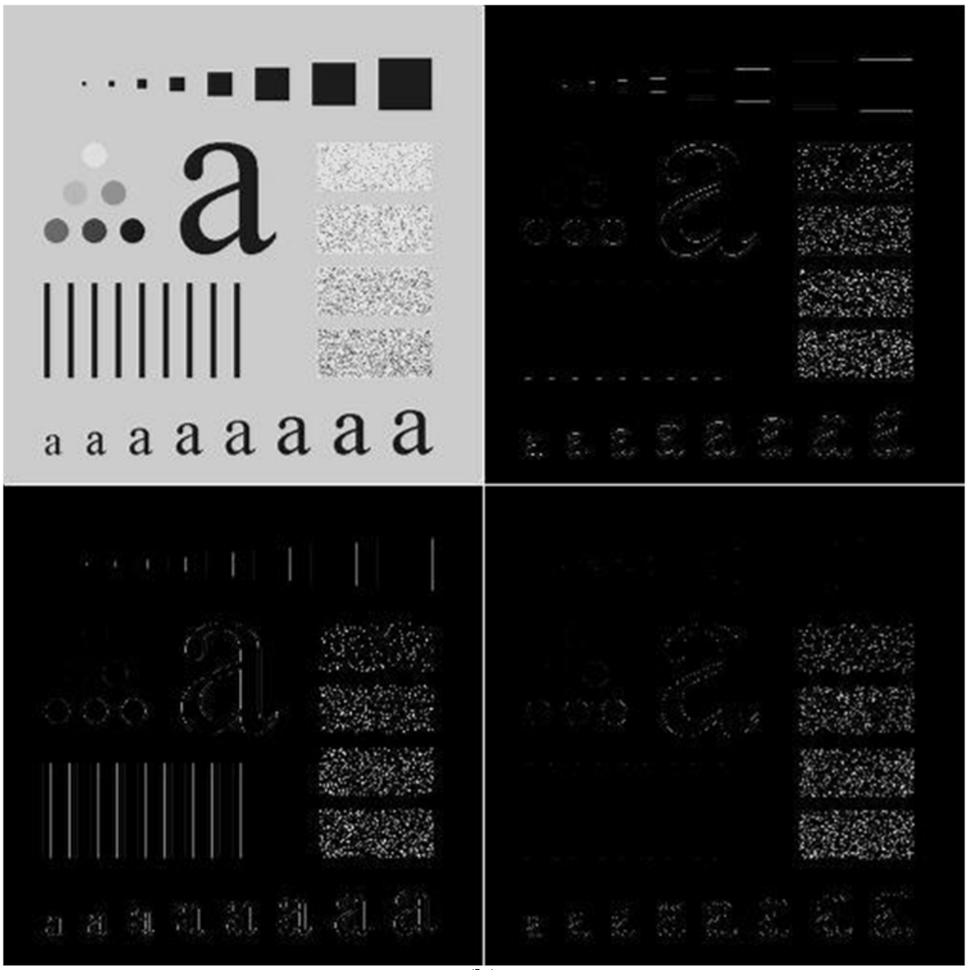
Most applications benefit from the statistical properties of the non-LL subbands (The Laplacian distribution of the wavelet coefficients in these subbands).

<u>Wavelet-based edge detection</u>

The next figure shows a gray image and its wavelet transform for one-level of decomposition.



(a)



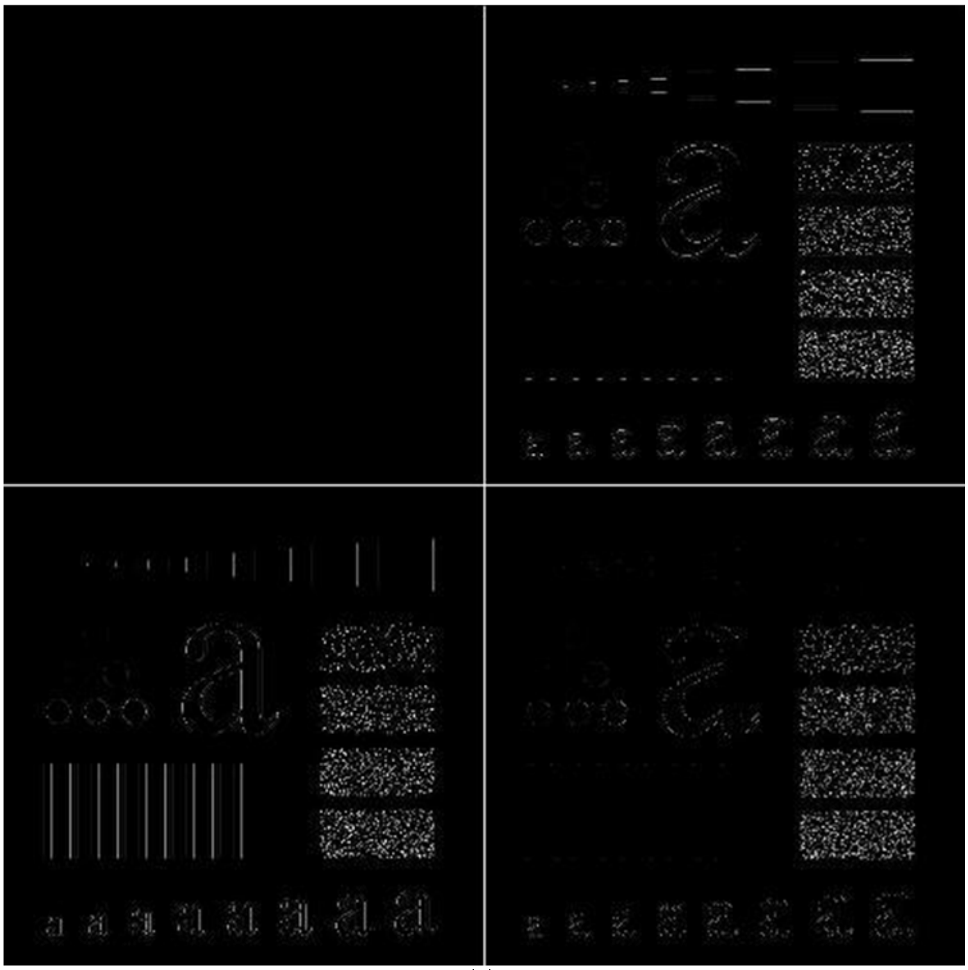
(b)

Figure 9.8 (a) gray image. (b) its one-level wavelet transform

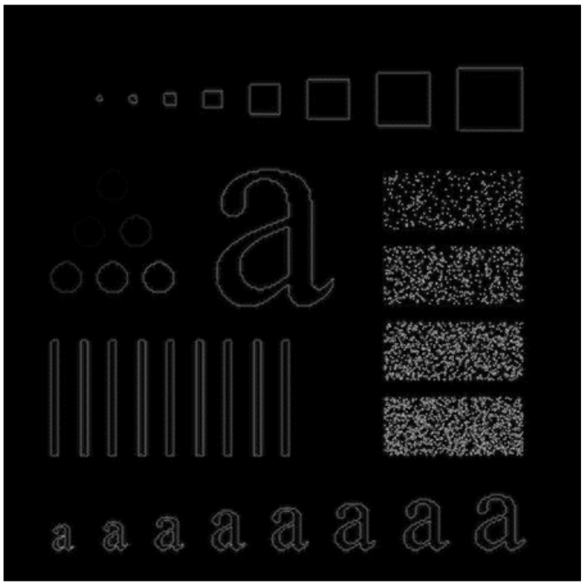
Note the horizontal edges of the original image are present in the HL subband of the upper-right quadrant of the Figure above. The vertical edges of the image can be similarly identified in the LH subband of the lower-left quadrant.

To combine this information into a single edge image, we simply zero the LL subband of the transform, compute the inverse transform, and take the absolute value.

The next Figure shows the modified transform and resulting edge image.



(a)



(b)

Figure 9.9 (a) transform modified by zeroing the LL subband. (b) resulted edge image

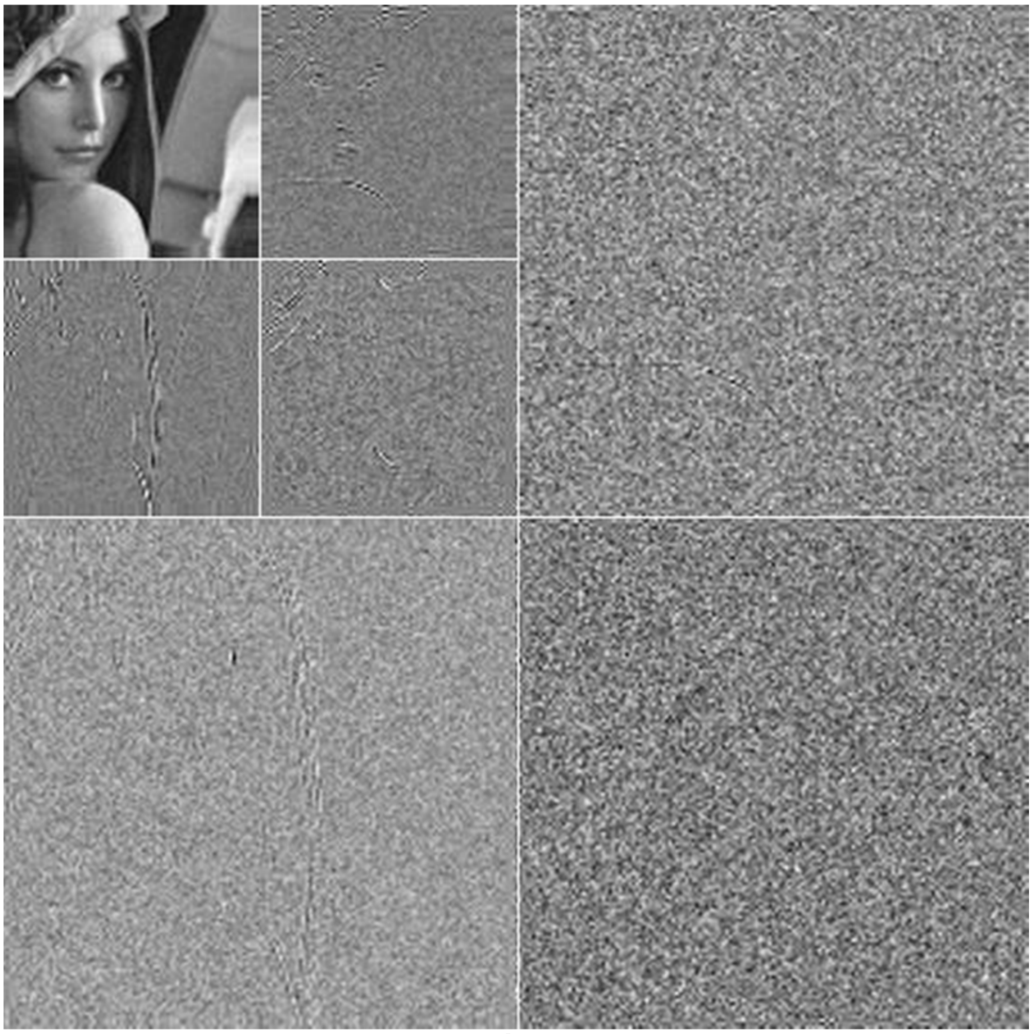
Wavelet-based image denoising

The general wavelet-based procedure for denoising the image is as follows:

1. Choose a wavelet filter (e.g. Haar, symlet, etc…) and number of levels for the decomposition. Then compute the 2D-DWT of the noisy image.
2. Threshold the non-LL subbands.
3. Perform the inverse wavelet transform on the original approximation LL-subband and the modified non-LL subbands.

The next figure shows a noisy image and its wavelet transform for two-levels of decomposition.

(a)



(b)

Figure 9.10 (a) noisy image. (b) its two-level wavelet transform

Now we threshold all the non-LL subbands at both decomposition levels by 85. Then we perform the inverse wavelet transform on the LL-subband and the modified (i.e. thresholded) non-LL subbands to obtain the denoised image shown in the next figure.

Figure 9.11 denoised image generated by thresholding all non-LL subbands by 85

In the image above, we can see the following:

- Noise Reduction.
- Loss of quality at the image edges.

The loss of edge detail can be reduced by zeroing the non-LL subbands at the first decomposition level and only the HH-subband at the second level. Then we apply the inverse transform to obtain the denoised image in the figure below.

Figure 9.12 denoised image generated by zeroing the non-LL subbands